Planarizing Gadgets for 2D Minimally Rigid Graphs Do Not Exist*

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Abstract

Recognizing *minimally rigid graphs* in 2D, also called *Laman graphs*, is a fundamental problem. The known algorithms for *planar* graphs are faster than those for general graphs. A common approach to transfer a complexity bound for a graph problem from the planar case to the general case is to use *planarizing gadgets*. We show unconditionally that planarizing gadgets for minimally rigid graphs do not exist.

Minimally rigid graphs are also called $(3,2)$-*tight* graphs. We generalize the result to the class of (k,l)-tight graphs that also captures other interesting graph classes like the graphs whose edges can be partitioned into two spanning trees.

By the same technique, we further show that planarizing gadgets do not exist for the problem of testing independence in the 2D *hyperconnectivity matroid*, a matroid that is closely related to rigidity, maximally recoverable tensor codes and low-rank matrix completion.

1 Introduction

A central question in *Graph Rigidity* is what properties make a graph *rigid*, i.e. what combinatorial properties of a graph G assure that almost all realizations of the graph as a *bar-and-joint framework* do not admit non-trivial, continuous deformations. Graph rigidity has applications in engineering and computational biology but it is also central problem in theoretical computer science because it yields an instance of the *polynomial identity testing (PIT)* problem [RW19].

Defining the problems. A graph $G = (V, E)$ with $|V| = n$ vertices and $|E| = m$ edges is *minimally rigid*, if the removal of any edge makes the graph non-rigid. Laman and Geiringer [Lam70, Pol27] independently showed that G is minimally rigid iff $m = 2n - 3$

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and every $S \subseteq V$, where $|S| \geq 2$, induces a subgraph in G with at most $2|S|-3$ many edges, i.e., $|E[S]| \leq 2|S|-3$, where $E[S]$ are the edges of G induced by S. In the literature, minimally rigid graphs are also called *Laman graphs*.

More generally, for parameters k, l, a (multi-) graph G is (k,l)-*tight*, if $m = kn - l$ and

$$\forall S \subseteq V \ (|S| \geq 2) \quad |E[S]| \leq k|S| - l. \tag{1}$$

Hence, Laman graphs are $(2,3)$-tight graphs.

A graph G is (k,l)-*sparse* if it fulfills (1) and $m \leq kn - l$. Graph G is (k,l)-*spanning*, if it contains a (k,l)-tight subgraph that spans the entire vertex set. *Rigid graphs* are $(2,3)$-spanning graphs.

The study of (k,l)-sparse graphs can be traced back to Loréa [Lor79]. Apart from the connection to rigidity, the class of (k,l)-sparse/tight/spanning graphs captures several interesting graph classes. We give some examples.

- The (k,k)-tight graphs are the graphs where the edges can be partitioned into k edge-disjoint spanning trees. In case of (k,k)-sparse graphs, the edges can be partitioned into k edge-disjoint forests [NW61, Tut61].
- For $k \leq l < 2k$, the (k,l)-sparse graphs are precisely the $(k, l-k)$-arborescences [Haa02]. A (k,a)-*arborescence* is a graph where adding any a edges results in a graph whose edge set is the union of k edge-disjoint spanning trees.
- For rational parameters k, l, there are also applications in discrete geometry [Whi96].

A (k,l)-spanning graph $G = (V,E)$ defines a *matroid* for $l < 2k$ and large enough n [LS08]: The ground set is E. A set $I \subseteq E$ is independent, if (V,I) is (k,l)-sparse. The base sets are the (k,l)-tight subsets. For $k = 2, l = 3$, the underlying matroid is the 2D *rigidity matroid*. For a graph $G = (V,E)$ consider an embedding of the vertices $p : V \to \mathbb{R}^2$. Let $R(G,p)$ be a $|E| \times 2|V|$ matrix where the row for an edge uv is defined as follows:

$$\begin{array}{cccccccc} & & & u & & & v & \\ uv & (0 \ldots 0 & p(u) - p(v) & 0 \ldots 0 & p(v) - p(u) & 0 \ldots 0). \end{array}$$

For a *generic embedding* p[1], the 2D rigidity matroid $\mathcal{R}_2$ is defined as the row matroid of $R(G,p)$, i.e. a set of edges is independent in this matroid if the corresponding rows in $R(G,p)$ are independent.

Kalai [Kal85] introduced another matroid on the edges of a graph that is very similar to the rigidity matroid, the 2D *hyperconnectivity matroid*. Again, for a graph $G = (V,E)$ consider an embedding of the vertices $p : V \to \mathbb{R}^2$. Let $H(G,p)$ be a $|E| \times 2|V|$ matrix where the row for an edge uv is defined as follows:

$$\begin{array}{cccccccc} & & & u & & & v & \\ uv & (0 \ldots 0 & p(v) & 0 \ldots 0 & -p(u) & 0 \ldots 0). \end{array}$$

[1]Generic means that the points are algebraically independent, roughly one can think of a generic embedding as a random embedding.

For a generic embedding p, the 2D hyperconnectivity matroid $\mathcal{H}_2$ is defined as the row matroid of $H(G,p)$. Using methods from *tropical geometry*, Bernstein [Ber17] showed that a set of edges is independent in $\mathcal{H}_2$ if and only if it admits an acyclic orientation with no alternating closed trails.

The matroids $\mathcal{R}_2$ and $\mathcal{H}_2$ are closely related: When a set of edges is independent in $\mathcal{H}_2$ it is also independent in $\mathcal{R}_2$ [CRS23]. The converse is not true, for example, the edges of the complete bipartite graph $K_{3,3}$ are independent in $\mathcal{R}_2$ but dependent in $\mathcal{H}_2$.

The matroids $\mathcal{R}_2, \mathcal{H}_2$ are closely related to *low-rank matrix completion*: Let $G = (V,E)$ be a graph on $|V| = n$ vertices. Define a skew-symmetric $n \times n$ matrix S, where the entries corresponding to E have been filled with generic values, and the remaining entries with variables. Then E is independent in the hyperconnectivity matroid, if and only if the variable entries of S can be assigned such that the matrix has rank two [CRS23]. There are also connections to *maximally recoverable tensor codes*, see the exposition in [BDG+26] for a detailed overview.

Known complexity bounds. Whether a graph $G = (V,E)$ is sparse, tight, or spanning for parameters (k,l) can be decided in time $\mathcal{O}(n^2)$ [LS08], for $|V| = n$ and $l \leq 2k-1$.

Hendrickson observed that the decision problem for $(2,3)$-sparse graphs reduces to *bipartite matching* [Hen92], and this reduction works similar for (k,l)-sparse graphs. Bipartite matching is in NC [CGG+26], hence, we get the same bound for deciding (k,l)-sparsity. Note that it is open whether (k,l)-spanning graphs can be recognized in NC, even for more restricted graph classes like planar graphs.

With respect to space-bounded computation it is open whether (k,l)-tightness or sparsity can be solved in Logspace or in NL.

For planar graphs, there is an NL-algorithm for deciding (k,l)-tightness: One can reduce (k,l)-tightness to the problem of deciding whether a given flow network with fixed *demands* at the source and sink vertices admits a feasible flow. Moreover, this reduction preserves the planarity. Miller and Naor [MN95] solve this flow problem with fixed demands in NC when the input graph is planar. The algorithm by Miller and Naor can also be implemented in NL because it reduces to the problem of deciding whether a weighted directed graph admits a negative cycle. We give the details in the Appendix.

For recognizing minimally rigid graphs, i.e. $(2,3)$-tight graphs or Laman graphs, the best known sequential algorithm has a running time of $\mathcal{O}(n\sqrt{n \log n})$ [GW92]. Planar minimally rigid graphs have a geometric characterization via *pointed pseudo-triangulations* [Str00, HOR+05]. Using this geometric characterization, planar minimally rigid graphs can be recognized in $\mathcal{O}(n \log^3 n)$, thereby improving the current bound for the general case [RSS19].

Testing independence in $\mathcal{H}_2$ is in RNC and in NP, it is open whether it is in P. It is not even known to be in coNP. For planar graphs, the matroids $\mathcal{H}_2$ and $\mathcal{R}_2$ are equivalent [CRS23, FJW04]. Thus, in the planar case, the problems can be solved in NC

Our results and their motivation. Given any complexity bound for a graph problem in the planar case, an approach to transfer this result to the general case is a *planarizing*

gadget that locally replaces edge crossings from a given drawing of a graph while preserving a desired property. We give some examples.

- Recall that (k, l)-tight graphs can be recognized in NL for *planar* graphs. If there would be a planarizing gadget it would imply an NL-bound for the non-planar case as well.
- A graph with a linear number of edges can have a quadratic crossing number. However, for graphs with a linear crossing number, a planarizing gadget would transfer the $\mathcal{O}(n \log^3 n)$ time bound for recognizing planar minimally rigid graphs to the non-planar case. In fact, up to crossing number $n^{3/2-\varepsilon}$, for any $\varepsilon > 0$, the strategy via planarizing gadget would outperform the best known sequential algorithm.
- For the problem of testing independence in $\mathcal{H}_2$, a planarizing gadget would give a polynomial time algorithm because the matroids $\mathcal{H}_2$ and $\mathcal{R}_2$ are equivalent in the planar case [CRS23, FJW04].

Planarizing gadgets have been used to show that classical NP-hard graph problems, like 3-colorability and vertex cover, are still NP-hard in the planar case [GJS76]. Datta, Kulkarni, Limaye and Mahajan used planarizing gadgets in the context of computing the determinant of an adjacency matrix [DKLM07]. We show unconditionally, that such planarizing gadgets do not exist for (k, l)-tight graphs.

Theorem 1.1. *Planarizing gadgets do not exist for* (k, l)*-tight graphs, for* $k \geq 2$ *and* $0 \leq l \leq 2k - 1$.

Recall that minimally rigid graphs are precisely the $(2, 3)$-tight graphs.

Corollary 1.2. *Planarizing gadgets do not exist for minimally rigid graphs.*

The result holds similarly for (k, l)-sparse graphs.

Theorem 1.3. *Planarizing gadgets do not exist for* (k, l)*-sparse graphs, for* $k \geq 2$ *and* $0 \leq l \leq 2k - 1$.

Note that Theorem 1.3 is not a generalization of Theorem 1.1, the two statements are incomparable. This is because when we have given a (k, l)-tight graph G in Theorem 1.3, the graph G' obtained from G by replacing a crossing with a gadget may become (k, l)-sparse, whereas in Theorem 1.1, it has to stay tight.

Using the same proof techniques, we also show that there is no planarizing gadget for testing independence in $\mathcal{H}_2$.

Theorem 1.4. *Planarizing gadgets do not exist for testing independence in* $\mathcal{H}_2$.

The non-existence of planarizing gadgets was already known for *bipartite perfect matching* [GKM+16]. As discussed above, recognizing (k, l)-tight graphs reduces to bipartite matching. A reduction in the other direction is not known. Though this problem reduces to bipartite perfect matching, not all results from bipartite perfect matching carry

over to (k,l)-tight graphs directly. In particular, the unconditional non-existence result for planarizing gadgets for bipartite perfect matching by Gurjar, Messner, Straub and Thierauf [GKM+16] does not seem to carry over to (k,l)-tight graphs and new ideas are required to solve the problem.

2 Preliminaries

Let $G = (V, E)$ be a graph. For a set of vertices $S \subseteq V$, we denote the subgraph of G induced by S with $G[S]$. We denote edges in the graph $G[S]$ by $E[S]$, i.e.

$$E[S] = \{ (u,v) \in E \mid u,v \in S \}.$$

If the edge set of a graph G is not explicitly given a name, we denote it by E_G.

We use integers k and l for sparsity and tightness conditions of graphs throughout the paper.

2.1 Planarizing gadgets

Roughly, a *planarizing gadget* is a planar graph of constant size which is used to replace edge crossings in the drawing of any graph G like in Figure 1, while preserving a desired property. More formally, let Γ be a graph of constant size that has a planar embedding with at least four vertices a, b, c, d that appear on the outer face in the order a, c, b, d. Let G be a graph with a drawing in the plane where edges ab, cd are crossing. When we replace this crossing in the drawing of G by Γ as in Figure 1, we get a drawing of $G' = G - \{ab, cd\} + \Gamma$ where the number of crossings is decreased by one. The idea is to replace each crossing in G by a copy of Γ to make the graph planar. Let L be a class of graphs. Then, the graph Γ is called a *planarizing gadget for* L if we have

$$G \in L \Leftrightarrow G' \in L.$$

Note that there might be multiple crossings in a drawing of G. In particular, one edge

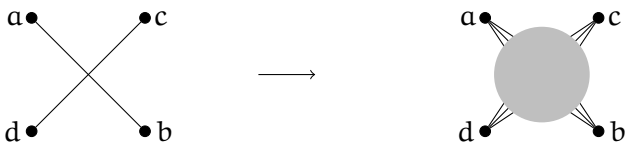


Figure 1: The crossing ab, cd is resolved by adding a planarizing gadget

could cross with multiple edges. We replace each crossing by a copy of Γ. When there are multiple crossings, one might need a more restricted gadget like the one in Figure 2. However, when we show that a planarizing gadget does not exist for the case that we only consider one edge crossing in the drawing, more restricted gadgets for multiple crossings

cannot exist as well. Throughout the paper we use G' to denote the graph $G - \{ab, cd\} + \Gamma$, i.e. the graph obtained by replacing the crossing edges $\{ab, cd\}$ with the gadget.

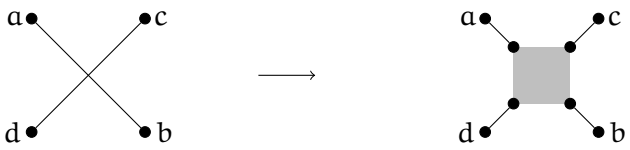


Figure 2: The crossing ab, cd is resolved by adding a more restricted gadget that also works for multiple crossings.

2.2 Connected components in sparse graphs

For the proof of Theorem 1.1 we need the fact that for some sparse graphs, induced subgraphs with sufficiently many edges are always connected.

Lemma 2.1. *Let $G = (V, E)$ be a (k, l)-sparse graph, where $1 \leq k \leq l \leq 2k - 1$. Then any subset $S \subseteq V$ with*

$$|E[S]| \geq k|S| - 2k + 1$$

induces a connected subgraph in G.

Proof. Assume S would induce a subgraph in G consisting of two disconnected components $C_1, C_2 \subseteq V$. Since G is (k, l)-tight we have

$$\begin{aligned} |E[S]| &= |E[C_1]| + |E[C_2]| \\ &\leq k|C_1| - l + k|C_2| - l \\ &= k|S| - 2l. \end{aligned}$$

Since we assume that $k \leq l$ this contradicts the assumption that $|E[S]| \geq k|S| - 2k + 1$. □

2.3 Replacing edges in (k, l)-tight graphs

We show that for a (k, l)-tight graph, edges can be replaced by a (planar) graph of constant size, while preserving the (k, l)-tight property.

Lemma 2.2. *Let $G = (V, E)$ be a graph with an edge $(u, v) \in E$ and let $0 \leq l \leq 2k - 1$. Let $\Omega = (V_\Omega, E_\Omega)$ be a $(k, 2k-1)$-tight graph with at least two vertices that we identify with u, v. Let $G^* = (V^*, E^*)$ be the graph obtained from G by removing edge (u, v) and taking the union of G and Ω by identifying nodes u and v in G and Ω. That is, $G^* = G - (u, v) + \Omega$ for short.*

Then G is (k, l)-tight iff G^* is (k, l)-tight.

Proof. Observe that

$$\begin{aligned} |V^*| &= |V| + |V_\Omega| - 2, \\ |E^*| &= |E| + |E_\Omega| - 1. \end{aligned} \tag{2}$$

Assume G is (k,l)-tight. We have

$$\begin{aligned} |E^*| &= k|V| - l - 1 + k|V_\Omega| - 2k + 1 \\ &= k|V^*| - l. \end{aligned}$$

Let $S \subseteq V^*$. When $u, v \notin S$, set S clearly has at most $k|S| - l$ many edges since G and Ω are (k,l)-sparse. Assume only one of $\{u,v\}$, say u, is contained in S. Then we have

$$\begin{aligned} |E^*[S]| &= |E[(S \setminus V_\Omega) \cup \{u\}]| + |E_\Omega[S \cap V_\Omega]| \\ &\leq k|(S \setminus V_\Omega) \cup \{u\}| - l + k|S \cap V_\Omega| - 2k + 1 \\ &\leq k|S| - l - k + 1 \\ &\leq k|S| - l. \end{aligned}$$

The last inequality follows since we assume that $k \geq 1$.

Now assume $u, v \in S$. Then we have

$$\begin{aligned} |E^*[S]| &= |E[(S \setminus V_\Omega) \cup \{u,v\}]| - 1 + |E_\Omega[S \cap V_\Omega]| \\ &\leq k|(S \setminus V_\Omega) \cup \{u,v\}| - l - 1 + k|S \cap V_\Omega| - 2k + 1 \\ &\leq k|S| - l. \end{aligned}$$

For the other direction, assume that G^* is (k,l)-tight. By Equation (2) we have

$$\begin{aligned} |E| &= |E^*| - |E_\Omega| + 1 \\ &= (k|V^*| - l) - (k|V_\Omega| - 2k + 1) + 1 \\ &= k|V| - l. \end{aligned}$$

Let $S \subseteq V$. Assume $u, v \in S$, otherwise S would clearly have at most $k|S| - l$ edges since G^* is (k,l)-tight. Let $T = S \cup V_\Omega$. Then we have

$$\begin{aligned} |E[S]| &= |E^*[T]| + 1 - |E_\Omega| \\ &\leq (k|T| - l) + 1 - (k|V_\Omega| - 2k + 1) \\ &= k|S| - l. \end{aligned}$$

□

When graph G has an edge crossing (a,b), (c,d), we put the gadget Γ to replace the crossing and obtain graph G'. By Lemma 2.2, we may assume that the gadget Γ does not have any of the edges within $\{a,b,c,d\}$, and the same will then hold for G'. Similarly it holds for G, except for (a,b), (c,d) that are in G.

2.4 Replacing edges in hyperconnected graphs

Analogously to Lemma 2.2 we show that for graphs whose edges are independent in $\mathcal{H}_2$, edges can be replaced by a (planar) graph of constant size, while preserving the independence.

Lemma 2.3. *Let* G *be a graph with an edge* $(u, v) \in E$*. Let* Ω *be a* K_4 *on the vertices* u, v, a, b *where we have removed the edge* (u, v)*. Let* G^* *be the graph obtained from* G *by removing edge* (u, v) *and taking the union of* G *and* Ω *by identifying nodes* u *and* v *in* G *and* Ω*. That is,* $G^* = G - (u, v) + \Omega$ *for short.*

Then G *is independent in* $\mathcal{H}_2$ *iff* G^* *is independent in* $\mathcal{H}_2$*.*

Proof. For one direction, assume that G is dependent in $\mathcal{H}_2$. We show that G^* is dependent as well. Let $p^* : V(G^*) \to \mathbb{R}^2$ be a generic embedding. Let p be the embedding p^* restricted to the vertices of G. Because G is dependent, there is a non-trivial dependency among the rows of the matrix $H(G, p)$

$$\alpha_1 r_1 + \dots \alpha_m r_m = 0.$$

If $\alpha_i = 0$, where i is the row corresponding to the edge (u, v), we would get a non-trivial dependency in $H(G^*, p^*)$ as well. So assume $\alpha_i \neq 0$. Recall that the gadget Ω is a K_4 where we have removed the edge (u, v). Since K_4 is a circuit in $\mathcal{H}_2$, the rows corresponding to the edges of Ω span the row corresponding to the edge (u, v). Thus, the graph G^* is also dependent.

For the reverse direction, assume that G is independent. We show that G^* is independent as well. Recall that G is independent iff there exists an acyclic orientation of its edges that has no alternating closed trail [Ber17]. Assume we have such an orientation for G where the edge uv is directed from u to v. We construct such an orientation in G^* as well. For that we keep all directed edges from G, except for (u, v), and direct the edges in Ω as $(u, a), (u, b), (a, b), (a, v), (b, v)$. Note that this orientation is acyclic in Ω and it has no alternating closed trails in Ω. To see that this orientation is acyclic in G^*, assume there would be a cycle. This cycle must then use edges from Ω and from G. In particular, this cycle must use the edges $(u, b), (b, v)$ or $(u, a), (a, b), (b, v)$ or $(u, a), (a, v)$. In all cases, we can replace the edges from Ω by the edge (u, v) to obtain a cycle in G, which is a contradiction.

To see that there is no alternating closed trail in G^* we can argue in a similar way. If there would be an alternating closed trail in G^*, this trail must use the edges on the path $\{u, b, a, v\}$. But then we obtain an alternating closed trail for G as well by replacing the edges from Ω with the edge (u, v). □

3 Proof of Theorem 1.1 and 1.3

We start with proving Theorem 1.1. At the end of the section, we explain the modifications of the proof to get Theorem 1.3 as well. We assume that $k \leq 3$, otherwise the statement

of the Theorem is trivial since a (k,l)-tight graph would have too many edges to be planar and hence there is no planarizing gadget. For the same reason, the cases $(3,l)$ are trivial for $l \leq 5$. Therefore, it only remains to show that no planarizing gadgets exist for $k = 2$ and $l = 0,1,2,3$. For our proof we assume for contradiction that a planarizing gadget exists. Clearly, any planarizing gadget must be (k,l)-sparse. Let $G = (V,E)$ be graph with a crossing $\{ab,cd\}$. We denote $G' = G - \{ab,cd\} + \Gamma$ as the graph in which we have replaced the crossing with the planarizing gadget. In Section 3.1 we first show that certain vertex sets induce subgraphs in the gadget which are *even more sparse*. In contrast to that, we show the existence of some subgraphs in the gadget which have a lower bound on number of edges (Section 3.2). Then we put everything together in Section 3.3 by constructing a subgraph in the gadget where the upper bound is smaller than the lower bound from the previous sections. Therefore we get a contradiction.

To derive the above properties of the gadget, we use different choices for the graph G, since the gadget must work for *all* graphs. For example, the gadget must also work for graphs where the end points of the crossing edges, a,b,c,d are connected to each other even after removing the edges ab,cd in G. Therefore in the planarizing gadget Γ, the vertices identified with a,b,c,d must lie on a common face, otherwise G' would have edge crossings (it is useful to think of Γ as lying inside a square bounding box with vertices a,c,b,d in clockwise order). This implies in particular that inside Γ there cannot be two vertex disjoint paths, one from a to b, and another from c to d, since Γ must be planar.

3.1 Upper bounds on the number of edges in the gadget

We first make an observation about the size of the gadget.

Lemma 3.1. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for (k,l)-tight graphs. Then we have*

$$|E_\Gamma| = k|V_\Gamma| - 4k + 2.$$

Proof. Let $G = (V,E)$ be a (k,l)-tight graph and $G' = (V',E') = G - \{ab,cd\} + \Gamma$ be the graph obtained from G when putting Γ. Note that G' is also (k,l)-tight, otherwise the gadget would not be valid. Hence we can compute

$$\begin{aligned} |E_\Gamma| &= |E'| - |E| + 2 \\ &= k|V'| - l - k|V| + l + 2 \\ &= k(|V_\Gamma| - 4) + 2 \\ &= k|V_\Gamma| - 4k + 2. \end{aligned}$$

□

Any subset of vertices S of Γ can induce at most $k|S| - l$ many edges, otherwise the graph G' would not be (k,l)-tight. When we consider subsets $S \subseteq V_\Gamma$ that contain the vertices a,b,c,d, we can strengthen the sparsity condition as follows.

Lemma 3.2. *Assume that* $\Gamma = (V_\Gamma, E_\Gamma)$ *is a planarizing gadget for* (k,l)*-tight graphs. Then, for every subset* $S \subseteq V_\Gamma$*, where* $a, b, c, d \in S$*, we have*

$$|E_\Gamma[S]| \leq k|S| - 4k + 2.$$

Proof. Let $G = (V, E)$ be a (k,l)-tight graph and $G' = (V', E') = G - \{ab, cd\} + \Gamma$ be the graph obtained from G when putting Γ. Let $S \subseteq V_\Gamma$ be a set of vertices such that $a, b, c, d \in S$. Since $G' = (V', E')$ must also be (k,l)-tight we have

$$\begin{aligned} |E_\Gamma[S]| &= |E'[S \cup V]| - (|E| - 2) \\ &\leq k|S \cup V| - l - |E| + 2 \\ &= k(|S| + |V| - 4) - l - (k|V| - l) + 2 \\ &= k|S| - 4k + 2. \end{aligned}$$

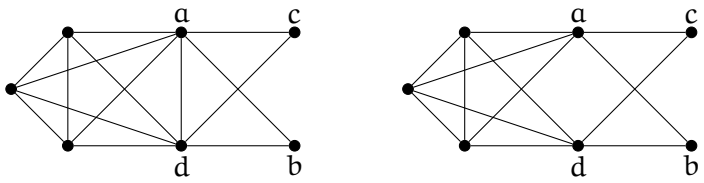
□

For $(2,0)$-tight graphs, we need a similar statement when S contains just two nodes from $\{a, b, c, d\}$, namely one from $\{a, b\}$ and one from $\{c, d\}$.

Lemma 3.3. *Assume that* $\Gamma = (V_\Gamma, E_\Gamma)$ *is a planarizing gadget for* $(2,0)$*-tight graphs. Then, for every subset* $S \subseteq V_\Gamma$*, where* $|\{a, b\} \cap S| = 1$ *and* $|\{c, d\} \cap S| = 1$*, we have*

$$|E_\Gamma[S]| \leq 2|S| - 4.$$

Proof. Consider the $(2,0)$-tight graph $G = (V, E)$ in Figure 3. Observe that $G \setminus \{b, c\}$ has 10 edges. Let $S \subseteq V_\Gamma$, where $a, d \in S$ and $b, c \notin S$. We have



$(2,0)$-tight $\qquad$ $(2,1)$-tight

Figure 3: The graphs used in the proofs of Lemma 3.3 and Lemma 3.4

.

$$\begin{aligned} |E_\Gamma[S]| &= |E'[S \cup V \setminus \{b, c\}]| - 10 \\ &\leq 2|S \cup V \setminus \{b, c\}| - 10 \\ &= 2(|S| + 3) - 10 \\ &= 2|S| - 4. \end{aligned}$$

By relabeling the vertices a, b, c, d in the graph from Figure 3 we obtain the bound for all cases from the statement. □

3.2 Lower bounds on the number of edges in the gadget

When the given graph G is not (k, l)-tight, graph G' should also be not (k, l)-tight. This means that G' must contain a subset of vertices that violates the (k, l)-sparsity condition. The following lemma is useful for such an argument.

Lemma 3.4. *Assume that* $\Gamma = (V_\Gamma, E_\Gamma)$ *is a planarizing gadget for* $(2, l)$*-tight graphs where* $l = 0, 1, 2, 3$*. Then there are two subsets* $S_1, S_2 \subseteq V_\Gamma$ *such that*

$$a, b \in S_1,\ c, d \notin S_1 \quad \textit{and} \quad |E_\Gamma[S_1]| \geq 2|S_1| - 3,$$
$$c, d \in S_2,\ a, b \notin S_2 \quad \textit{and} \quad |E_\Gamma[S_2]| \geq 2|S_2| - 3.$$

Moreover, the subgraphs $\Gamma[S_1], \Gamma[S_2]$ *contain a path from* a *to* b *and* c *to* d*, respectively.*

Proof. We first show the existence of the set S_1. The existence of S_2 follows by a similar argument when we change the labeling of the vertices in the crossing by mapping a, b, c, d to c, d, b, a, respectively. Consider Figure 4 where we have three graphs G, H_1, H_2. The graph

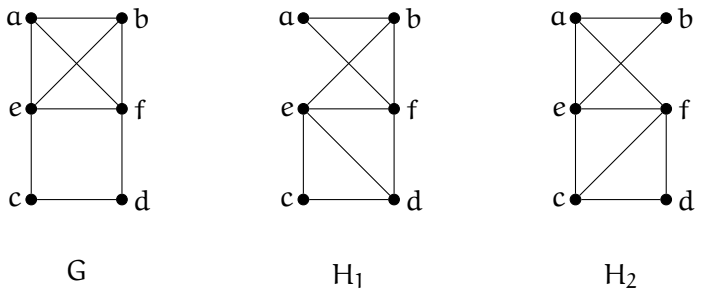


Figure 4: The graphs for Lemma 3.4 for $l = 3$. Graph G is not $(2, 3)$-tight, but H_1, H_2 are.

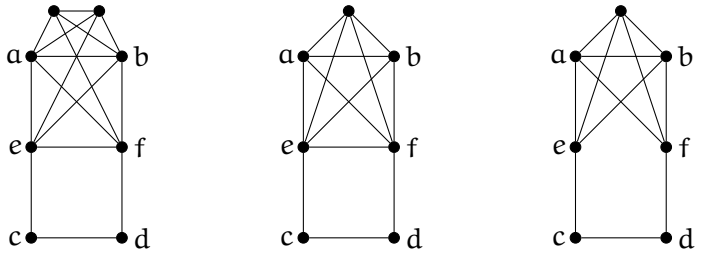
$G = (V, E)$ is a not $(2, 3)$-tight because the set $V - \{c, d\}$ violates the sparsity condition. When we move the edge (a, e) to (d, e) we obtain a new graph H_1 which is $(2, 3)$-tight. Analogously, we obtain a graph H_2 by moving the edge (b, f) to (c, f) which is also $(2, 3)$-tight.

We can construct similar graphs for the cases $l = 0, 1, 2$, see Figure 5.

Since G is not $(2, l)$-tight, the graph $G' = (V', E')$ where we have put the gadget, must also be not $(2, l)$-tight. Hence, there exists a set $T \subseteq V'$ that violates the $(2, l)$-sparsity condition,

$$|E'[T]| > 2|T| - l. \tag{3}$$

However, the graphs H_1' and H_2' where we put the gadget must both be $(2, l)$-tight and thus, the same set T must fulfill the $(2, l)$-sparsity condition in H_1', H_2'. That is, $|E_{H_i'}[T]| \leq 2|T| - l$, for $i = 1, 2$. The difference between G and H_1, H_2 is that one edge has moved. It follows



$l = 0$ $l = 1$ $l = 2$

Figure 5: The graphs G for Lemma 3.4 for $l = 0, 1, 2$. The corresponding graphs H_1 and H_2 are defined analogously as in the case $l = 3$ in Figure 4.

that $a, b \in T$ and $c, d \notin T$, otherwise the edge moves would not decrease the number of edges in $E_{H_1'}[T]$ and $E_{H_2'}[T]$.

Let $S_1 = T \cap V_\Gamma$ be the nodes of T inside the gadget. Recall that we defined Γ such that $a, b, c, d \in V_\Gamma$. Hence, we have $a, b \in S_1$ and $c, d \notin S_1$, as required by the lemma. Now we want to argue the lower bound on the number of edges induced by S_1.

Let $T_1 = T \setminus S_1 \cup \{a, b\}$ be the nodes of T outside the gadget, but including a, b. By Lemma 2.2, we may assume that $(a, b) \notin E_\Gamma$. Note that G becomes $(2, l)$-sparse when we remove the edge (a, b). Hence, $G'[T_1]$ must be $(2, l)$-sparse,

$$|E'[T_1]| \leq 2|T_1| - l. \tag{4}$$

By definition, we have $T = T_1 \cup S_1$. Since $(a, b) \notin E'$, from (3) and (4), we get

$$\begin{aligned} |E_\Gamma[S_1]| = |E'[S_1]| &= |E'[T]| - |E'[T_1]| \\ &\geq (2|T| - l + 1) - (2|T_1| - l) \\ &= 2|T| + 1 - 2|T_1| \\ &= 2|T| + 1 - 2(|T| - |S_1| + 2) \\ &= 2|S_1| - 3. \end{aligned}$$

It remains to argue the last part that the subgraph $\Gamma[S_1]$ contains a path from a to b. By Lemma 2.1, the subgraph $\Gamma[S_1]$ is connected for $l \geq 2$ and we are done.

For $l = 0, 1$, let G be the $(2, 0)$-tight or the $(2, 1)$-tight graph in Figure 3. Assume there is no path from a to b in S_1. Hence, we can partition S_1 into two components A, B such that $a \in A$ and $b \in B$, and $\Gamma[A]$ and $\Gamma[B]$ are disconnected. Since G is $(2, l)$-tight, G' must also be $(2, l)$-tight.

Choose $X = A \cup V - \{b, c\}$. For both graphs G in Figure 3, for $l = 0, 1$, we have that $|X| = |A| + |V - \{b, c\}| - 1 = |A| + 4$, because A and V have one node in common, a. Also, without nodes b, c, the graphs have $|E[V - \{b, c\}]| = 10 - l$ edges. Since X must fulfill the sparsity condition in G', we get

$$\begin{aligned} |E_\Gamma[A]| &= |E'[X]| - |E[V - \{b, c\}]| \\ &= |E'[X]| - (10 - l) \\ &\leq 2|X| - l - 10 + l \\ &= 2(|A| + 4) - 10 \\ &= 2|A| - 2. \end{aligned}$$

When we change the labeling of the vertices in the example by mapping a to b we get the same bound for $|E_\Gamma[B]|$ as well. But then we get a contradiction:

$$\begin{aligned} 2|S_1| - 3 &\leq |E_\Gamma[S_1]| \\ &= |E_\Gamma[A]| + |E_\Gamma[B]| \\ &\leq 2|A| - 2 + 2|B| - 2 \\ &= 2|S_1| - 4. \end{aligned}$$

□

3.3 Getting a contradiction from upper and lower bound

In this section, we prove Theorem 1.1 by showing that the upper bound from Lemma 3.2 contradicts the lower bound from Lemma 3.4: Consider the two subsets S_1, S_2 from Lemma 3.4. Note that a, b, c, d are contained in $S_1 \cup S_2$. We show that $S_1 \cup S_2$ violates Lemma 3.2. We distinguish different cases depending on the cardinality of $S_1 \cap S_2$.

Case 1. $S_1 \cap S_2 = \emptyset$. By Lemma 3.4 we know that the subgraphs induced by S_1, S_2 contain a path P_{ab} from a to b and a path P_{cd} from c to d, respectively. A planar embedding of the gadget graph Γ where the vertices a, c, d, b appear on the outerface in that order looks like the one in Figure 6. Since we assume $S_1 \cap S_2 = \emptyset$, the sets S_1, S_2 have no common vertices and thus, the paths P_{ab}, P_{cd} must cross with each other. Therefore, the case $S_1 \cap S_2 = \emptyset$ does not emerge as the gadget would not be planarizing.

Case 2. $|S_1 \cap S_2| = 1$. Then we have

$$\begin{aligned} |E_\Gamma[S_1 \cup S_2]| &\geq |E_\Gamma[S_1]| + |E_\Gamma[S_2]| \\ &\geq 2|S_1| - 3 + 2|S_2| - 3 \qquad (5) \\ &= 2(|S_1 \cup S_2| + 1) - 6 \\ &= 2|S_1 \cup S_2| - 4 \end{aligned}$$

Inequality (5) follows from Lemma 3.4 and contradicts Lemma 3.2.

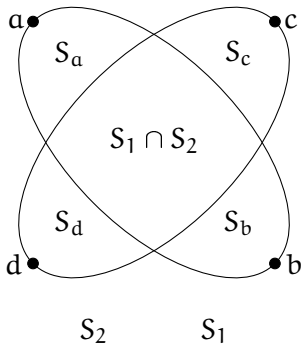


Figure 6: Drawing of the gadget graph Γ with the sets S_1, S_2 from Lemma 3.4.

Case 3. $|S_1 \cap S_2| \geq 2$. Then we have

$$|E_\Gamma[S_1 \cup S_2]| \geq |E_\Gamma[S_1]| + |E_\Gamma[S_2]| - |E_\Gamma[S_1 \cap S_2]| \tag{6}$$
$$\geq 2|S_1| - 3 + 2|S_2| - 3 - (2|S_1 \cap S_2| - l) \tag{7}$$
$$= 2|S_1 \cup S_2| - 6 + l. \tag{8}$$

Inequality (7) follows from Lemma 3.4 and the fact that $\Gamma[S_1 \cap S_2]$ must be (k, l)-sparse. Inequality (8) contradicts Lemma 3.2 when $l \geq 1$.

The case $l = 0$ is more elaborate. Since Γ is a planarizing gadget it must admit a planar embedding where the vertices a, c, b, d appear on the outer-face in that order, as in Figure 6. Observe that in this embedding, the set S_2 naturally partitions the set S_1 into three parts $S_a, S_1 \cap S_2, S_b$ such that $a \in S_a, b \in S_b$. Note that there cannot be any edges in the gadget with one end point in S_a and other end point in S_b. Vice versa, S_1 partitions S_2 into $S_c, S_1 \cap S_2, S_d$. We give an upper bound on the number of edges in $E_\Gamma[S_1]$ that contradicts Lemma 3.4.

For $x \in \{a, b, c, d\}$ we define

$$S'_x = S_x \cup (S_1 \cap S_2).$$

and we let $z_x \geq 0$ be an integer such that

$$|E_\Gamma[S'_x]| = 2|S'_x| - z_x.$$

Note that we can assume that there are no edges going from S_a to S_c, S_d or from S_b to S_c, S_d, otherwise (6) would be larger and thus we get a contradiction. For the same reason, we can assume $|E_\Gamma[S_1 \cap S_2]| = 2|S_1 \cap S_2|$ by (7). Thus, for the two integers $z_a, z_d \geq 0$, we

have

$$\begin{aligned}|E_\Gamma[S'_a \cup S'_d]| &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_d]| - |E_\Gamma[S_1 \cap S_2]| \\ &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_d]| - 2|S_1 \cap S_2| \\ &= 2|S'_a| - z_a + 2|S'_d| - z_d - 2|S_1 \cap S_2| \\ &= 2|S'_a \cup S'_d| - (z_a + z_d).\end{aligned}$$

By Lemma 3.3, we also have $|E_\Gamma[S'_a \cup S'_d]| \leq 2|S'_a \cup S'_d| - 4$. Thus, we get

$$z_a + z_d \geq 4.$$

When we apply the same arguments to the pairs (S'_a, S'_c), (S'_b, S'_d), (S'_b, S'_c) we obtain the constraints

$$\begin{aligned}z_a + z_c &\geq 4, \\ z_b + z_d &\geq 4, \\ z_b + z_c &\geq 4.\end{aligned}$$

It follows that $z_a + z_b \geq 4$ or $z_c + z_d \geq 4$. Assume $z_a + z_b \geq 4$. Since there are no edges between S_a, S_b, we have

$$\begin{aligned}|E_\Gamma[S_1]| &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_b]| - |E_\Gamma[S_1 \cap S_2]| \\ &= 2|S'_a| - z_a + 2|S'_b| - z_b - 2|S_1 \cap S_2| \\ &= 2|S_1| - (z_a + z_b) \\ &\leq 2|S_1| - 4\end{aligned}$$

But from Lemma 3.4, we must have $|E_\Gamma[S_1]| \geq 2|S_1| - 3$, which gives a contradiction. This finishes the proof of Theorem 1.1.

Theorem 1.3 claims the nonexistence of planarizing gadgets for sparse graphs instead of tight graphs. Lemma 2.2 can be slightly modified so that edges can be replaced with a constant size graph while preserving sparsity. When we modify the proof of Theorem 1.1 for sparsity, the equality in Lemma 3.1 becomes an upper bound and the other arguments still work the same way.

4 Proof of Theorem 1.4

The proof works along the lines of the proof for Theorem 1.1. Assume there exists a planarizing gadget Γ for testing independence in $\mathcal{H}_2$. Recall that any graph whose edges are independent in $\mathcal{H}_2$ must be $(2,3)$-sparse. Hence, we can transfer Lemma 3.2 that gives an upper bound on the number of edges in Γ to hyperconnectivity.

Lemma 4.1. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for testing independence in $\mathcal{H}_2$. Then, for every subset $S \subseteq V_\Gamma$, where $a, b, c, d \in S$, we have*

$$|E_\Gamma[S]| \leq 2|S| - 6.$$

To get a lower bound as in Lemma 3.4 requires a new argument. We state the Lemma for 2D hyperconnectivity.

Lemma 4.2. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for testing independence in $\mathcal{H}_2$. Then there are two subsets $S_1, S_2 \subseteq V_\Gamma$ such that*

$$a, b \in S_1,\ c, d \notin S_1 \quad \textit{and} \quad |E_\Gamma[S_1]| \geq 2|S_1| - 3,$$
$$c, d \in S_2,\ a, b \notin S_2 \quad \textit{and} \quad |E_\Gamma[S_2]| \geq 2|S_2| - 3.$$

Moreover, the subgraphs $\Gamma[S_1], \Gamma[S_2]$ contain a path from a to b and c to d, respectively.

Proof. Like in the proof of Lemma 3.4 it suffices to show the existence of S_1. The existence of S_2 then follows by relabeling the vertices. Let G, H_1, H_2 be the graphs from Figure 4. The graph G is not independent in $\mathcal{H}_2$ because it contains a K_4 on the vertices a, b, e, f. The graphs H_1, H_2 are both independent in $\mathcal{H}_2$. Now consider a drawing of G like in Figure 7. In this drawing, there is exactly one crossing $\{ab, cd\}$. Hence, the graph G' where we put the planarizing gadget becomes planar.

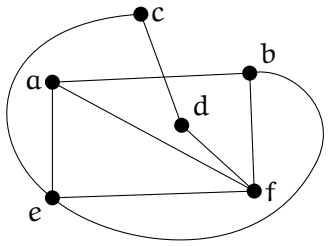


Figure 7: Redrawing of the graph G for Lemma 4.2.

Recall that a planar graph is independent in $\mathcal{H}_2$ if and only if it is $(2,3)$-sparse [CRS23, FJW04]. It follows that there must exist a set T in G' that violates the $(2,3)$-sparsity condition. Then we can choose $S_1 = T \cap V(\Gamma)$. The rest of the proof works exactly like the proof of Lemma 3.4. At the point where we apply Lemma 2.2 we apply Lemma 2.3 instead. □

With the upper and lower bounds from Lemmas 4.1 and 4.2, we can continue as in Section 3.3 to get a contraction for the existence of a planarizing gadget. This proves Theorem 1.4.

5 Discussion

We showed the nonexistence of planarizing gadgets for sparse and tight graphs. We leave open the case of spanning graphs. Note that our arguments do not work directly when

we replace *tight* with *spanning* in Theorem 1.1 since the size of the gadget can become arbitrarily large in this case.

Acknowledgments

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References

[BDG+26] Joshua Brakensiek, Manik Dhar, Jiyang Gao, Sivakanth Gopi, and Matt Larson. Rigidity Matroids and Linear Algebraic Matroids with Applications to Matrix Completion and Tensor Codes. *Combinatorica*, 46(2), March 2026.

[Ber17] Daniel Irving Bernstein. Completion of tree metrics and rank 2 matrices. *Linear Algebra and its Applications*, 533:1–13, 2017.

[CGG+26] Abhranil Chatterjee, Sumanta Ghosh, Rohit Gurjar Gurjar, Roshan Raj, and Thomas Thierauf. Bipartite Matching is in NC. Research Report TR26-100, ECCC, 2026.

[CRS23] Luis Crespo Ruiz and Francisco Santos. Bar-and-joint rigidity on the moment curve coincides with cofactor rigidity on a conic. *Combinatorial Theory*, 3(1), March 2023.

[DKLM07] Samir Datta, Raghav Kulkarni, Nutan Limaye, and Meena Mahajan. Planarity, Determinants, Permanents, and (Unique) Matchings. In *Computer Science – Theory and Applications*, pages 115–126. Springer Berlin Heidelberg, 2007.

[FJW04] Zsolt Fekete, Tibor Jordán, and Walter Whiteley. An inductive construction for plane laman graphs via vertex splitting. In Susanne Albers and Tomasz Radzik, editors, *Algorithms – ESA 2004*, pages 299–310, Berlin, Heidelberg, 2004. Springer Berlin Heidelberg.

[GJS76] M.R. Garey, D.S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237–267, 1976.

[GKM+16] Rohit Gurjar, Arpita Korwar, Jochen Messner, Simon Straub, and Thomas Thierauf. Planarizing Gadgets for Perfect Matching Do Not Exist. *ACM Trans. Comput. Theory*, 8(4), July 2016.

[GW92] Harold N. Gabow and Herbert H. Westermann. Forests, frames, and games: Algorithms for matroid sums and applications. *Algorithmica*, 7(1–6):465–497, June 1992.

[Haa02] Ruth Haas. Characterizations of arboricity of graphs. *Ars Comb.*, 63, 2002.

[Hen92] Bruce Hendrickson. Conditions for unique graph realizations. *SIAM Journal on Computing*, 21(1):65–84, February 1992.

[HLS97] Christoph M. Hoffmann, Andrew Lomonosov, and Meera Sitharam. Finding solvable subsets of constraint graphs. In Gert Smolka, editor, *Principles and Practice of Constraint Programming-CP97*, pages 463–477, Berlin, Heidelberg, 1997. Springer Berlin Heidelberg.

[HLS98] Christoph M. Hoffmann, Andrew Lomonosov, and Meera Sitharam. *Geometric Constraint Decomposition*, pages 170–195. Springer Berlin Heidelberg, Berlin, Heidelberg, 1998.

[HOR+05] Ruth Haas, David Orden, Günter Rote, Francisco Santos, Brigitte Servatius, Hermann Servatius, Diane Souvaine, Ileana Streinu, and Walter Whiteley. Planar minimally rigid graphs and pseudo-triangulations. *Computational Geometry*, 2005.

[Imm88] Neil Immerman. Nondeterministic space is closed under complementation. *SIAM Journal on Computing*, 17(5):935–938, 1988.

[Kal85] Gil Kalai. Hyperconnectivity of graphs. *Graphs and Combinatorics*, 1(1):65–79, December 1985.

[Lam70] Gerard Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering Mathematics*, 4(4):331–340, October 1970.

[Lor79] M. Lorea. On matroidal families. *Discrete Mathematics*, 28(1):103–106, 1979.

[LS08] Audrey Lee and Ileana Streinu. Pebble game algorithms and sparse graphs. *Discrete Mathematics*, 308(8):1425–1437, 2008. 3rd European Conference on Combinatorics.

[MN95] Gary L. Miller and Joseph Naor. Flow in planar graphs with multiple sources and sinks. *SIAM Journal on Computing*, 24(5):1002–1017, October 1995.

[NW61] Crispin St. J. A. Nash-Williams. Edge-Disjoint Spanning Trees of Finite Graphs. *Journal of the London Mathematical Society*, 36:445–450, 1961.

[Pol27] H. Pollaczek-Geiringer. Über die Gliederung ebener Fachwerke. *Journal of Applied Mathematics and Mechanics*, 7(1):58–72, January 1927.

[Rei08] Omer Reingold. Undirected connectivity in log-space. *Journal of the ACM*, 55(4), 2008.

[RSS19] Jonathan Rollin, Lena Schlipf, and André Schulz. Recognizing Planar Laman Graphs. In *27th Annual European Symposium on Algorithms (ESA)*, volume 144 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 79:1–79:12. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2019.

[RW19] Orit E. Raz and Avi Wigderson. *Subspace Arrangements, Graph Rigidity and Derandomization Through Submodular Optimization*, pages 377–415. Springer Berlin Heidelberg, 2019.

[Str00] Ileana Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *41st Symposium on Foundations of Computer Science (FOCS)*, pages 443–453. IEEE Computer Society, 2000.

[Sze88] Róbert Szelepcsényi. The method of forced enumeration for nondeterministic automata. *Acta Informatica*, 26(3):279–284, November 1988.

[Tut61] William T. Tutte. On the Problem of Decomposing a Graph into n Connected Factors. *Journal of the London Mathematical Society*, 36:221–230, 1961.

[Whi96] Walter Whiteley. Some Matroids from Discrete Applied Geometry. In *Matroid Theory*, pages 171–311. American Mathematical Society, 1996.

A Reducing (k, l)-sparsity to max flow with multiple sources and sinks

We describe a reduction from (k, l)-sparsity to *max flow with multiple sources and sinks (MSMS max flow)*. The reduction is the first step to show that planar tightness is in NL. There are similar reductions in the literature, see e.g. [HLS97, HLS98]. Still the precise setting here needs some adaptions.

Max flow with multiple sources and sinks is defined as follows.

Definition A.1 (MSMS max flow). *We are given a tuple* (G, S, T, c)*, where*

- $G = (V, E)$ *is a directed graph,*
- $S, T \subseteq V$ *are disjoint sets of* sources *and* sinks*, respectively. The vertices in* S *have no incoming edges and the vertices in* T *have no outgoing edges in* G*,*
- $c : E \to \mathbb{Z}_{\geq 0}$ *is the* capacity function*.*

A flow *in the graph is a function* $f : E \to \mathbb{R}_{\geq 0}$*. For* $v \in V$*, the* net outgoing flow *from* v *is denoted by* f_v*,*

$$f_v = \sum_{(v,w)\in E} f(v, w) - \sum_{(w,v)\in E} f(w, v).$$

A flow f *is* feasible*, if*

- $\forall e \in E \quad f(e) \leq c(e)$*,*
- $\forall v \in V - (S \cup T) \quad f_v = 0$*.*

Given a feasible flow f*, its* value *is defined as* $|f| = \sum_{s\in S} f_s$*. The objective of the problem is to find the maximum value of a feasable flow for* (G, S, T, c)*.*

We show a reduction from sparseness to MSMS flows.

Lemma A.2. *Let* $G = (V, E)$ *be graph with* $|V| = n$ *and* $|E| = m$*, and* $0 \leq l \leq 2k - 1$*. Deciding whether* G *is* (k, l)*-sparse reduces to solving* m *many MSMS flow problems.*

Moreover, when G *is planar, has bounded genus or has bounded treewidth, these flow problems will have the corresponding property too.*

Proof. We construct a MSMS flow problem (H, S, T, c) as follows. Graph $H = (V_H, E_H)$ has vertices $V_H = S \cup E \cup V \cup T$, where S is a copy of E and T is a copy of V. That is, for $E = \{e_1, e_2, \ldots, e_m\}$, let $S = \{s_1, s_2, \ldots, s_m\}$, where $s_i \in S$ is associated with $e_i \in E$. Similarly, for $V = \{v_1, v_2, \ldots, v_n\}$, let $T = \{t_1, t_2, \ldots, t_n\}$, where $t_i \in T$ is associated with $v_i \in V$.

The edges of H are

$$E_H = E_1 \cup E_2 \cup E_3,$$

where

- E_1 are the S-E edges, $E_1 = \{(s_i, e_i) \mid i \in [m]\}$ with capacities $c(s_i, e_i) = 1$.
- E_2 are the E-V edges, $E_2 = \{(e, v) \mid e \in E,\ v \in e\}$ with capacities $c(e, v) = \alpha = m+l+1$. The value of α is chosen large enough such that the E_2-edges can never appear in an S-T-cut of value at most $m + l$.
- E_3 are the V-T edges, $E_3 = \{(v_i, t_i) \mid i \in [n]\}$ with capacities $c(v_i, t_i) = k$.

Observe that H is constructed from G by subdividing edges or adding vertices of degree one. Hence, the construction preserves the genus, in particular planarity, and the treewidth of the input graph G.

Flow problem (H, S, T, c) serves as the base from which we define m variants that we actually use in the reduction. For each edge $e \in E$, we define essentially the same flow problem (H, S, T, c_e), only the capacity function is changed for one edge: Let $s \in S$ be the copy of $e \in E$. Then we define c_e to be same as c except for edge (s, e) where we set

$$c_e(s, e) = l + 1.$$

The construction of (H, S, T, c_e) is shown in Figure 8.

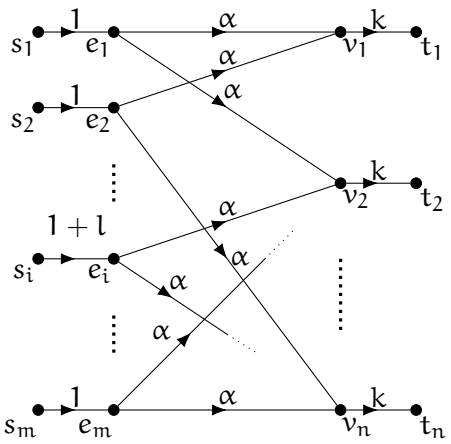


Figure 8: The flow problem (H, S, T, c_e) as described in the reduction with $e_i = e$.

We argue that G is (k, l)-sparse iff the flow problems (H, S, T, c_e) have max flow $m + l$, for all $e \in E$. By the *max-flow min-cut theorem*, we can argue via (S, T)-cuts in H. A set $C \subseteq V_H$ is a (S, T)-*cut*, if $S \subseteq C$ and $C \cap T = \emptyset$. The *capacity of cut* C for capacity function c_e is

$$c_e(C) = \sum_{u \in C,\, v \in V_H \setminus C} c_e(u, v).$$

We will show that

$$G \text{ not } (k, l)\text{-sparse} \iff \exists e \in E \ \ \exists C \subseteq V_H \ (S, T)\text{-cut} \quad c_e(C) < m + l.$$

Assume first that G is not (k,l)-sparse, i.e. there is a subset $X \subseteq V$ such that $|X| \geq 2$ and $|E_G[X]| > k|X| - l$. Choose an edge $e \in E[X]$ and consider the cut $C = S \cup E_G[X] \cup X \subseteq V_H$ in H for capacity function c_e. The cut edges for C are the edges from S to $E \setminus E_G[X]$ of capacity 1 and the edges from X to T of capacity k. Hence, we have

$$\begin{aligned} c_e(C) &= m - |E_G[X]| + k|X| \\ &< k|X| + m - (k|X| - l) \\ &= m + l. \end{aligned}$$

For the backward direction, assume that for an edge $e \in E$ there is a flow problem (H, S, T, c_e) with a cut C, where $c_e(C) < m + l$. We list some properties of cut C.

- $C \neq S$, because $c_e(S) = m + l$.
- C does not cut through edges from E_2 going from E to V as they all have capacity $\alpha = m + l + 1$ and thus, the cut would be too large already from one such edge.

Therefore we have $C = S \cup F \cup X$ for some non-empty sets $F \subseteq E$ and $X \subseteq V$, such that $X' = V_G(F) \subseteq X$, where $V_G(F)$ are the endpoints of the edges in F. The latter point is because of the second item above. Note that $|X'| \geq 2$ since $|F| \geq 1$.

We have two cases depending on whether the edge e is in F or not.

Case 1. $e \in F$. Then we have

$$\begin{aligned} c_e(C) &= |E \setminus F| + k|X| \\ &= m - |F| + k|X|. \end{aligned}$$

Hence, $c_e(C) < m + l$ is equivalent to $|F| > k|X| - l$. It follows that G is not (k,l)-sparse because

$$|E_G[X']| \geq |F| > k|X| - l \geq k|X'| - l.$$

The last inequality follows because $X' \subseteq X$.

Case 2. $e \notin F$. Then we have $c_e(C) = m - |F| + l + k|X|$. Similarly as in the first case, we conclude that $|E_G[X']| > k|X'| \geq k|X'| - l$. □

B Max flow with fixed demands in NL

One can extend a MSMS flow problem by fixing *demands* at the sources and sinks, i.e. for every source (sink) we specify a number such that the outgoing flow (incoming flow) must be equal to that number. Then we obtain a decision problem: Does the given flow network with fixed demands admit a feasible flow. This decision problem can be solved in NL when the input graph is planar:

Lemma B.1. *Whether a given flow network with fixed demands admits a feasible flow can be decided in* NL

Proof. Miller and Naor solve this problem efficiently in parallel, they even compute a flow if it exists [MN95]. We give a sketch of their algorithm:

1. Find any flow function f' in G that satisfies demands for all vertices of G.
   This is done by constructing a spanning tree T of G (the underlying undirected graph). An edge e of G is assigned a flow value of of zero if it is not in T. If e is in T then it is assigned a flow value based on sum of demands of the vertices lying in the subtree of T attached to head of e.

2. Construct the residual flow graph $G' = G - f$. The graph G' will be a circulation graph (i.e. every vertex of G' has demand 0).

3. Compute a feasible circulation f'' in G'.
   To do this, first we need to check if the dual graph of G', G^*, has no negative cycle. If a negative cycle is detected then we terminate the algorithm and output that no feasible flow satisfying the demands exists for G. If G^* has no negative cycles, then we choose any arbitrary vertex f_0 of G^* and compute distances from f_0 to every other vertex of G^* (this corresponds to computing shortest distances of faces of G' from a particular face f_0). The flow value of an edge e in the circulation f'' is simply the difference between the shortest distances values of the faces adjacent to e.

4. Return the flow $f = f' + f''$ as output.

Except for step three, all other steps can be done in Logspace using transducers. A spanning tree of an undirected graph can becomputed in Logspace using Reingold's algorithm [Rei08]. For the decision version, we do not need Step 4, and also do not need to compute distances in Step 3. Therefore the algorithm can be seen as a Logspace reduction to detecting negative cycles in a graph with polynomialy bounded weights. By the Immerman-Szelepcsényi Theorem [Imm88, Sze88] we have that coNL = NL. Hence, it is enough to see that the existence of a negative cycle can be decided in NL. This is trivial because one can just guess a negative cycle. □

Note that for (k, l)-tightness we can add demands to the flow problems in the proof of Lemma A.2 by taking the sum of the capacities on the outgoing and incoming arcs as demands of the sources and sinks, respectively. Thus, by Lemma A.2 and B.1, planar (k, l)-tightness is in NL.

Corollary B.2. *Recognizing planar* (k, l)*-tight graphs is in* NL.