

Planarizing Gadgets for (k, l) -tight Graphs Do Not Exist*

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Abstract

The problem of *recognizing (k, l) -tight graphs* is a fundamental problem that has close connections to well studied problems like graph rigidity. The problem is better understood for planar graphs as compared to general graphs. For example, deterministic NC-algorithms for the problem are known for planar graphs, but no such algorithm is known for general graphs. A common approach to reduce a graph problem to the planar case is to use *planarizing gadgets*. Our main contribution is to show that, unconditionally, planarizing gadgets for the problem of *recognizing (k, l) -tight graphs* do not exist.

1 Introduction

Let $G = (V, E)$ be a (multi-) graph with $|V| = n$ vertices and $|E| = m$ edges. Graph G is (k, l) -*sparse*, for integers k, l , if every subset $S \subseteq V$, where $|S| \geq 2$, induces a subgraph in G with at most $k|S| - l$ many edges. A graph G is (k, l) -*tight*, if G is (k, l) -sparse and $m = kn - l$. Graph G is (k, l) -*spanning*, if it contains a (k, l) -tight subgraph that spans the entire vertex set.

The class of (k, l) -sparse/tight/spanning graphs captures several interesting graph classes. We give some examples.

- The $(2, 3)$ -tight graphs are the *minimally rigid graphs* in 2D, also called *Laman graphs*. The $(2, 3)$ -spanning graphs are the *rigid graphs* in 2D [Lam70, Pol27].
- The (k, k) -tight graphs are the graphs where the edges can be partitioned into k edge-disjoint spanning trees. In case of (k, k) -sparse graphs, the edges can be partitioned into k edge-disjoint forests [NW61, Tut61].
- For $k \leq l < 2k$, the (k, l) -sparse graphs are precisely the $(k, l-k)$ -arborescences [Haa02]. A (k, a) -*arborescence* is a graph where adding any a edges results in a graph whose edge set is the union of k edge-disjoint spanning trees.

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A (k, l) -spanning graph $G = (V, E)$ defines a *matroid* for $l < 2k$ and large enough n [LS08]: The ground set is E . A set $I \subseteq E$ is independent, if (V, I) is (k, l) -sparse. The base sets are the (k, l) -tight subsets.

For rational parameters k, l , there are also applications in discrete geometry [Whi96].

Complexity. Whether a graph $G = (V, E)$ is sparse, tight, or spanning for parameters (k, l) can be decided in time $\mathcal{O}(n^2)$ [LS08], for $|V| = n$ and $l \leq 2k - 1$.

With respect to the *parallel complexity*, it is open whether (k, l) -sparse, tight, or spanning graphs can be recognized in NC, or even Logspace. Note that the problem of deciding if a graph is (k, l) -sparse is slightly more general than deciding if a graph is (k, l) -tight¹. Hendrickson observed that the decision problem for $(2, 3)$ -sparse graphs reduces to *bipartite perfect matching* [Hen92], and this reduction works similar for (k, l) -sparse graphs. Since bipartite perfect matching is in RNC [MVV87] and in quasi-NC [FGT16], we get the same bounds, RNC and quasi-NC, for deciding (k, l) -sparsity.

For *planar graphs*, we get NC-algorithms in some cases. For example, whether a planar graph G is $(2, 3)$ -tight can be decided in NC, in fact, even when G is only *one-crossing-minor-free* [GRT26].

Hendrickson's reduction from (k, l) -sparsity to bipartite perfect matching does not preserve planarity. However, there is a reduction from (k, l) -sparsity to *s-t max flow* (see [HLS97, HLS98]), that can be tweaked to reduce (k, l) -sparsity to *max flow with multiple sources and sinks* such that planarity is preserved. For completeness, we give this reduction in the appendix Section A. Sankowski [San18] reduces max flow to *weighted perfect matching* which can be solved in NC when the input graph is planar or has bounded genus [AV20]. Sankowski's reduction preserves planarity and also the genus of the graph. Therefore, we can decide (k, l) -sparsity in NC for planar graphs and also for bounded-genus graphs. Moreover, max flow is in NC when the input graph has bounded treewidth [HKNR98]. Hence, (k, l) -sparsity is also in NC for bounded-treewidth graphs. We describe this in more detail in Appendix A.

For recognizing planar (k, l) -tight graphs, there is also an alternative NC algorithm. For tightness one can modify the flow reduction by fixing *demands* at the source and sink vertices. Miller and Naor [MN95] solve this more restricted flow problem in NC when the input graph is planar.

Note that it is open whether (k, l) -spanning graphs can be recognized in NC, even for planar graphs.

Our contribution. By the results for planar graphs mentioned above, an obvious idea to bring the decision problem for (k, l) -tight graphs down to NC is to reduce it to the planar case. A common approach to such a reduction is a *planarizing gadget* that locally replaces edge crossings from a given drawing of a graph while preserving a desired property, like

¹To see this, observe that if we are given an oracle to decide (k, l) -sparsity, then for any graph, we can decide if it is (k, l) -tight by just checking if $m = kn - l$. However if we have a graph where $m \neq kn - l$, then even with an oracle for deciding (k, l) -tightness there is no obvious efficient way to rule out (k, l) -sparsity.

(k, l) -sparsity. Planarizing gadgets have been used to show that classical NP-hard graph problems, like 3-colorability and vertex cover, are still NP-hard in the planar case [GJS76]. Datta, Kulkarni, Limaye and Mahajan used planarizing gadgets in the context of computing the determinant of an adjacency matrix [DKLM07]. We show unconditionally, that such planarizing gadgets do not exist for (k, l) -tight graphs.

Theorem 1.1. *Planarizing gadgets do not exist for (k, l) -tight graphs, for $k \geq 2$ and $0 \leq l \leq 2k - 1$.*

The result from Theorem 1.1 holds analogously for *bipartite perfect matching* [GKM⁺16]. The question whether bipartite perfect matching is in NC is still open and much work has been done to consider different variations of the problem in restricted graph classes or other complexity classes closely related to NC [FGT16, MVV87, Kul06, DKR09, AV20, MV00, ST17, AM25, San18]. As discussed above, recognizing (k, l) -tight graphs reduces to bipartite perfect matching. A reduction in the other direction is not known. Hence, (k, l) -tightness might be easier to solve. Though this problem reduces to bipartite perfect matching, not all results from bipartite perfect matching carry over to (k, l) -tight graphs directly. In particular, the unconditional non-existence result for planarizing gadgets for bipartite perfect matching by Gurjar, Messner, Straub and Thierauf [GKM⁺16] does not seem to carry over to (k, l) -tight graphs and new ideas are required to solve the problem.

Also in terms of sequential time complexity our result might be interesting: For recognizing $(2, 3)$ -tight graphs, the best known sequential algorithm has a running time of $\mathcal{O}(n\sqrt{n \log n} + m)$ [GW92]. Planar $(2, 3)$ -tight graphs (a.k.a. *planar Laman graphs*) have a geometric characterization via *pointed pseudo-triangulations* [Str00, HOR⁺05]. Using this geometric characterization, planar $(2, 3)$ -tight graphs can be recognized in $\mathcal{O}(n \log^3 n)$, thereby improving the time complexity for the general case [RSS19]. Our non-existence result for planarizing gadgets poses an obstacle in improving the sequential running time for the general case by reducing it to the planar case.

2 Preliminaries

Let $G = (V, E)$ be a graph. For a set of vertices $S \subseteq V$, we denote the subgraph of G induced by S with $G[S]$. We denote edges in the graph $G[S]$ by $E[S]$, i.e.

$$E[S] = \{(u, v) \in E \mid u, v \in S\}.$$

If the edge set of a graph G is not explicitly given a name, we denote it by E_G .

We use integers k and l for sparsity and tightness conditions of graphs throughout the paper.

2.1 Planarizing gadgets

Roughly, a *planarizing gadget* is a planar graph of constant size which is used to replace edge crossings in the drawing of any graph G like in Figure 1, while preserving a desired

property. More formally, let Γ be a graph of constant size that has a planar embedding with at least four vertices a, b, c, d that appear on the outer face in the order a, c, b, d . Let G be a graph with a drawing in the plane where edges ab, cd are crossing. When we replace this crossing in the drawing of G by Γ as in Figure 1, we get a drawing of $G' = G - \{ab, cd\} + \Gamma$ where the number of crossings is decreased by one. The idea is to replace each crossing in G by a copy of Γ to make the graph planar. Let L be a class of graphs. Then, the graph Γ is called a *planarizing gadget* for L if we have

$$G \in L \Leftrightarrow G' \in L.$$

Note that there might be multiple crossings in a drawing of G . In particular, one edge

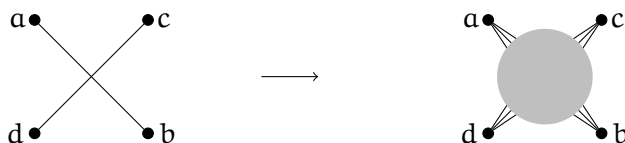


Figure 1: The crossing ab, cd is resolved by adding a planarizing gadget

could cross with multiple edges. We replace each crossing by a copy of Γ . When there are multiple crossings, one might need a more restricted gadget like the one in Figure 2. However, when we show that a planarizing gadget does not exist for the case that we only consider one edge crossing in the drawing, more restricted gadgets for multiple crossings cannot exist as well. Throughout the paper we use G' to denote the graph $G - \{ab, cd\} + \Gamma$, i.e. the graph obtained by replacing the crossing edges $\{ab, cd\}$ with the gadget.

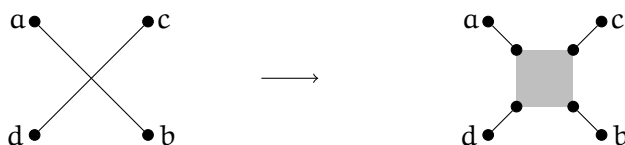


Figure 2: The crossing ab, cd is resolved by adding a more restricted gadget that also works for multiple crossings.

2.2 Connected components in sparse graphs

For the proof of our main result we need the fact that for some sparse graphs, induced subgraphs with sufficiently many edges are always connected.

Lemma 2.1. *Let $G = (V, E)$ be a (k, l) -sparse graph, where $1 \leq k \leq l \leq 2k - 1$. Then any subset $S \subseteq V$ with*

$$|E[S]| \geq k|S| - 2k + 1$$

induces a connected subgraph in G .

Proof. Assume S would induce a subgraph in G consisting of two disconnected components $C_1, C_2 \subseteq V$. Since G is (k, l) -tight we have

$$\begin{aligned} |E[S]| &= |E[C_1]| + |E[C_2]| \\ &\leq k|C_1| - l + k|C_2| - l \\ &= k|S| - 2l. \end{aligned}$$

Since we assume that $k \leq l$ this contradicts the assumption that $|E[S]| \geq k|S| - 2k + 1$. \square

2.3 Replacing edges

We show that for a (k, l) -tight graph, edges can be replaced by a (planar) graph of constant size, while preserving the (k, l) -tight property.

Lemma 2.2. *Let $G = (V, E)$ be a graph with an edge $(u, v) \in E$ and let $0 \leq l \leq 2k - 1$. Let $\Omega = (V_\Omega, E_\Omega)$ be a $(k, 2k - 1)$ -tight graph with at least two vertices that we identify with u, v . Let $G^* = (V^*, E^*)$ be the graph obtained from G by removing edge (u, v) and taking the union of G and Ω by identifying nodes u and v in G and Ω . That is, $G^* = G - (u, v) + \Omega$ for short.*

Then G is (k, l) -tight iff G^ is (k, l) -tight.*

Proof. Observe that

$$\begin{aligned} |V^*| &= |V| + |V_\Omega| - 2, \\ |E^*| &= |E| + |E_\Omega| - 1. \end{aligned} \tag{1}$$

Assume G is (k, l) -tight. We have

$$\begin{aligned} |E^*| &= k|V| - l - 1 + k|V_\Omega| - 2k + 1 \\ &= k|V^*| - l. \end{aligned}$$

Let $S \subseteq V^*$. When $u, v \notin S$, set S clearly has at most $k|S| - l$ many edges since G and Ω are (k, l) -sparse. Assume only one of $\{u, v\}$, say u , is contained in S . Then we have

$$\begin{aligned} |E^*[S]| &= |E[(S \setminus V_\Omega) \cup \{u\}]| + |E_\Omega[S \cap V_\Omega]| \\ &\leq k|(S \setminus V_\Omega) \cup \{u\}| - l + k|S \cap V_\Omega| - 2k + 1 \\ &\leq k|S| - l - k + 1 \\ &\leq k|S| - l. \end{aligned}$$

The last inequality follows since we assume that $k \geq 1$.

Now assume $u, v \in S$. Then we have

$$\begin{aligned} |E^*[S]| &= |E[(S \setminus V_\Omega) \cup \{u, v\}]| - 1 + |E_\Omega[S \cap V_\Omega]| \\ &\leq k|(S \setminus V_\Omega) \cup \{u, v\}| - l - 1 + k|S \cap V_\Omega| - 2k + 1 \\ &\leq k|S| - l. \end{aligned}$$

For the other direction, assume that G^* is (k, l) -tight. By Equation (1) we have

$$\begin{aligned} |E| &= |E^*| - |E_\Omega| + 1 \\ &= (k|V^*| - l) - (k|V_\Omega| - 2k + 1) + 1 \\ &= k|V| - l. \end{aligned}$$

Let $S \subseteq V$. Assume $u, v \in S$, otherwise S would clearly have at most $k|S| - l$ edges since G^* is (k, l) -tight. Let $T = S \cup V_\Omega$. Then we have

$$\begin{aligned} |E[S]| &= |E^*[T]| + 1 - |E_\Omega| \\ &\leq (k|T| - l) + 1 - (k|V_\Omega| - 2k + 1) \\ &= k|S| - l. \end{aligned}$$

□

When graph G has an edge crossing $(a, b), (c, d)$, we put the gadget Γ to replace the crossing and obtain graph G' . By Lemma 2.2, we may assume that the gadget Γ does not have any of the edges within $\{a, b, c, d\}$, and the same will then hold for G' . Similarly it holds for G , except for $(a, b), (c, d)$ that are in G .

3 Proof of Theorem 1.1

We assume that $k \leq 3$, otherwise the statement of the Theorem is trivial since a (k, l) -tight graph would have too many edges to be planar and hence there is no planarizing gadget. For the same reason, the cases $(3, l)$ are trivial for $l \leq 5$. Therefore, it only remains to show that no planarizing gadgets exist for $k = 2$ and $l = 0, 1, 2, 3$. For our proof we assume for contradiction that a planarizing gadget exists. Clearly, any planarizing gadget must be (k, l) -sparse. Let $G = (V, E)$ be graph with a crossing $\{ab, cd\}$. We denote $G' = G - \{ab, cd\} + \Gamma$ as the graph in which we have replaced the crossing with the planarizing gadget. In Section 3.1 we first show that certain vertex sets induce subgraphs in the gadget which are *even more sparse*. In contrast to that, we show the existence of some subgraphs in the gadget which have a lower bound on number of edges (Section 3.2). Then we put everything together in Section 3.3 by constructing a subgraph in the gadget where the upper bound is smaller than the lower bound from the previous sections. Therefore we get a contradiction.

To derive the above properties of the gadget, we use different choices for the graph G , since the gadget must work for *all* graphs. For example, the gadget must also work for graphs where the end points of the crossing edges, a, b, c, d are connected to each other even after removing the edges ab, cd in G . Therefore in the planarizing gadget Γ , the vertices identified with a, b, c, d must lie on a common face, otherwise G' would have edge crossings (it is useful to think of Γ as lying inside a square bounding box with vertices a, c, b, d in clockwise order). This implies in particular that inside Γ there cannot be two vertex disjoint paths, one from a to b , and another from c to d , since Γ must be planar.

3.1 Upper bounds on the number of edges in the gadget

We first make an observation about the size of the gadget.

Lemma 3.1. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for (k, l) -tight graphs. Then we have*

$$|E_\Gamma| = k|V_\Gamma| - 4k + 2.$$

Proof. Let $G = (V, E)$ be a (k, l) -tight graph and $G' = (V', E') = G - \{ab, cd\} + \Gamma$ be the graph obtained from G when putting Γ . Note that G' is also (k, l) -tight, otherwise the gadget would not be valid. Hence we can compute

$$\begin{aligned} |E_\Gamma| &= |E'| - |E| + 2 \\ &= k|V'| - l - k|V| + l + 2 \\ &= k(|V_\Gamma| - 4) + 2 \\ &= k|V_\Gamma| - 4k + 2. \end{aligned}$$

□

Any subset of vertices S of Γ can induce at most $k|S| - l$ many edges, otherwise the graph G' would not be (k, l) -tight. When we consider subsets $S \subseteq V_\Gamma$ that contain the vertices a, b, c, d , we can strengthen the sparsity condition as follows.

Lemma 3.2. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for (k, l) -tight graphs. Then, for every subset $S \subseteq V_\Gamma$, where $a, b, c, d \in S$, we have*

$$|E_\Gamma[S]| \leq k|S| - 4k + 2.$$

Proof. Let $G = (V, E)$ be a (k, l) -tight graph and $G' = (V', E') = G - \{ab, cd\} + \Gamma$ be the graph obtained from G when putting Γ . Let $S \subseteq V_\Gamma$ be a set of vertices such that $a, b, c, d \in S$. Since $G' = (V', E')$ must also be (k, l) -tight we have

$$\begin{aligned} |E_\Gamma[S]| &= |E'[S \cup V]| - (|E| - 2) \\ &\leq k|S \cup V| - l - |E| + 2 \\ &= k(|S| + |V| - 4) - l - (k|V| - l) + 2 \\ &= k|S| - 4k + 2. \end{aligned}$$

□

For $(2, 0)$ -tight graphs, we need a similar statement when S contains just two nodes from $\{a, b, c, d\}$, namely one from $\{a, b\}$ and one from $\{c, d\}$.

Lemma 3.3. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for $(2, 0)$ -tight graphs. Then, for every subset $S \subseteq V_\Gamma$, where $|\{a, b\} \cap S| = 1$ and $|\{c, d\} \cap S| = 1$, we have*

$$|E_\Gamma[S]| \leq 2|S| - 4.$$

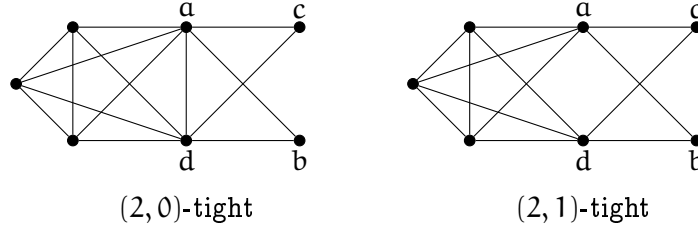


Figure 3: The graphs used in the proofs of Lemma 3.3 and Lemma 3.4

Proof. Consider the $(2,0)$ -tight graph $G = (V, E)$ in Figure 3. Observe that $G \setminus \{b, c\}$ has 10 edges. Let $S \subseteq V_\Gamma$, where $a, d \in S$ and $b, c \notin S$. We have

$$\begin{aligned}
 |E_\Gamma[S]| &= |E'[S \cup V \setminus \{b, c\}]| - 10 \\
 &\leq 2|S \cup V \setminus \{b, c\}| - 10 \\
 &= 2(|S| + 3) - 10 \\
 &= 2|S| - 4.
 \end{aligned}$$

By relabeling the vertices a, b, c, d in the graph from Figure 3 we obtain the bound for all cases from the statement. \square

3.2 Lower bounds on the number of edges in the gadget

When the given graph G is not (k, l) -tight, graph G' should also be not (k, l) -tight. This means that G' must contain a subset of vertices that violates the (k, l) -sparsity condition. The following lemma is useful for such an argument.

Lemma 3.4. *Assume that $\Gamma = (V_\Gamma, E_\Gamma)$ is a planarizing gadget for $(2, l)$ -tight graphs where $l = 0, 1, 2, 3$. Then there are two subsets $S_1, S_2 \subseteq V_\Gamma$ such that*

$$\begin{aligned}
 a, b \in S_1, \quad c, d \notin S_1 \quad \text{and} \quad |E_\Gamma[S_1]| &\geq 2|S_1| - 3, \\
 c, d \in S_2, \quad a, b \notin S_2 \quad \text{and} \quad |E_\Gamma[S_2]| &\geq 2|S_2| - 3.
 \end{aligned}$$

Moreover, the subgraphs $\Gamma[S_1], \Gamma[S_2]$ contain a path from a to b and c to d , respectively.

Proof. We first show the existence of the set S_1 . The existence of S_2 follows by a similar argument when we change the labeling of the vertices in the crossing by mapping a, b, c, d to c, d, b, a , respectively. Consider Figure 4 where we have three graphs G, H_1, H_2 . The graph $G = (V, E)$ is a not $(2,3)$ -tight because the set $V - \{c, d\}$ violates the sparsity condition. When we move the edge (a, e) to (d, e) we obtain a new graph H_1 which is $(2,3)$ -tight.

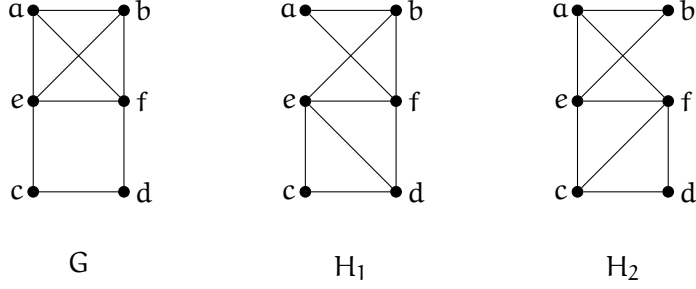


Figure 4: The graphs for Lemma 3.4 for $l = 3$. Graph G is not $(2, 3)$ -tight, but H_1, H_2 are.

Analogously, we obtain a graph H_2 by moving the edge (b, f) to (c, f) which is also $(2, 3)$ -tight.

We can construct similar graphs for the cases $l = 0, 1, 2$, see Figure 5.

Since G is not $(2, l)$ -tight, the graph $G' = (V', E')$ where we have put the gadget, must also be not $(2, l)$ -tight. Hence, there exists a set $T \subseteq V'$ that violates the $(2, l)$ -sparsity condition,

$$|E'[T]| > 2|T| - l. \quad (2)$$

However, the graphs H_1' and H_2' where we put the gadget must both be $(2, l)$ -tight and thus, the same set T must fulfill the $(2, l)$ -sparsity condition in H_1', H_2' . That is, $|E_{H_i'}[T]| \leq 2|T| - l$, for $i = 1, 2$. The difference between G and H_1, H_2 is that one edge has moved. It follows that $a, b \in T$ and $c, d \notin T$, otherwise the edge moves would not decrease the number of edges in $E_{H_1'}[T]$ and $E_{H_2'}[T]$.

Let $S_1 = T \cap V_\Gamma$ be the nodes of T inside the gadget. Recall that we defined Γ such that $a, b, c, d \in V_\Gamma$. Hence, we have $a, b \in S_1$ and $c, d \notin S_1$, as required by the lemma. Now we want to argue the lower bound on the number of edges induced by S_1 .

Let $T_1 = T \setminus S_1 \cup \{a, b\}$ be the nodes of T outside the gadget, but including a, b . By Lemma 2.2, we may assume that $(a, b) \notin E_\Gamma$. Note that G becomes $(2, l)$ -sparse when we remove the edge (a, b) . Hence, $G'[T_1]$ must be $(2, l)$ -sparse,

$$|E'[T_1]| \leq 2|T_1| - l. \quad (3)$$

By definition, we have $T = T_1 \cup S_1$. Since $(a, b) \notin E'$, from (2) and (3), we get

$$\begin{aligned} |E_\Gamma[S_1]| &= |E'[S_1]| = |E'[T]| - |E'[T_1]| \\ &\geq (2|T| - l + 1) - (2|T_1| - l) \\ &= 2|T| + 1 - 2|T_1| \\ &= 2|T| + 1 - 2(|T| - |S_1| + 2) \\ &= 2|S_1| - 3. \end{aligned}$$

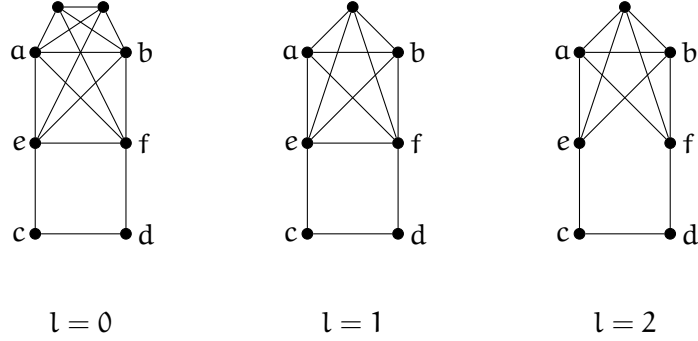


Figure 5: The graphs G for Lemma 3.4 for $l = 0, 1, 2$. The corresponding graphs H_1 and H_2 are defined analogously as in the case $l = 3$ in Figure 4.

It remains to argue the last part that the subgraph $\Gamma[S_1]$ contains a path from a to b . By Lemma 2.1, the subgraph $\Gamma[S_1]$ is connected for $l \geq 2$ and we are done.

For $l = 0, 1$, let G be the $(2, 0)$ -tight or the $(2, 1)$ -tight graph in Figure 3. Assume there is no path from a to b in S_1 . Hence, we can partition S_1 into two components A, B such that $a \in A$ and $b \in B$, and $\Gamma[A]$ and $\Gamma[B]$ are disconnected. Since G is $(2, l)$ -tight, G' must also be $(2, l)$ -tight.

Choose $X = A \cup V - \{b, c\}$. For both graphs G in Figure 3, for $l = 0, 1$, we have that $|X| = |A| + |V - \{b, c\}| - 1 = |A| + 4$, because A and V have one node in common, a . Also, without nodes b, c , the graphs have $|E[V - \{b, c\}]| = 10 - l$ edges. Since X must fulfill the sparsity condition in G' , we get

$$\begin{aligned}
 |E_\Gamma[A]| &= |E'[X]| - |E[V - \{b, c\}]| \\
 &= |E'[X]| - (10 - l) \\
 &\leq 2|X| - l - 10 + l \\
 &= 2(|A| + 4) - 10 \\
 &= 2|A| - 2.
 \end{aligned}$$

When we change the labeling of the vertices in the example by mapping a to b we get the same bound for $|E_\Gamma[B]|$ as well. But then we get a contradiction:

$$\begin{aligned}
 2|S_1| - 3 &\leq |E_\Gamma[S_1]| \\
 &= |E_\Gamma[A]| + |E_\Gamma[B]| \\
 &\leq 2|A| - 2 + 2|B| - 2 \\
 &= 2|S_1| - 4.
 \end{aligned}$$

□

3.3 Getting a contradiction from upper and lower bound

In this section, we prove Theorem 1.1 by showing that the upper bound from Lemma 3.2 contradicts the lower bound from Lemma 3.4: Consider the two subsets S_1, S_2 from Lemma 3.4. Note that a, b, c, d are contained in $S_1 \cup S_2$. We show that $S_1 \cup S_2$ violates Lemma 3.2. We distinguish different cases depending on the cardinality of $S_1 \cap S_2$.

Case 1. $S_1 \cap S_2 = \emptyset$. By Lemma 3.4 we know that the subgraphs induced by S_1, S_2 contain a path P_{ab} from a to b and a path P_{cd} from c to d , respectively. A planar embedding of the gadget graph Γ where the vertices a, c, d, b appear on the outerface in that order looks like the one in Figure 6. Since we assume $S_1 \cap S_2 = \emptyset$, the sets S_1, S_2 have no common vertices and thus, the paths P_{ab}, P_{cd} must cross with each other. Therefore, the case $S_1 \cap S_2 = \emptyset$ does not emerge as the gadget would not be planarizing.

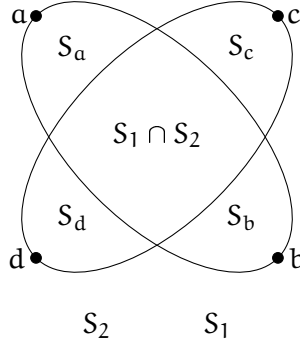


Figure 6: Drawing of the gadget graph Γ with the sets S_1, S_2 from Lemma 3.4.

Case 2. $|S_1 \cap S_2| = 1$. Then we have

$$\begin{aligned}
 |E_{\Gamma}[S_1 \cup S_2]| &\geq |E_{\Gamma}[S_1]| + |E_{\Gamma}[S_2]| \\
 &\geq 2|S_1| - 3 + 2|S_2| - 3 \\
 &= 2(|S_1 \cup S_2| + 1) - 6 \\
 &= 2|S_1 \cup S_2| - 4
 \end{aligned} \tag{4}$$

Inequality (4) follows from Lemma 3.4 and contradicts Lemma 3.2.

Case 3. $|S_1 \cap S_2| \geq 2$. Then we have

$$|E_{\Gamma}[S_1 \cup S_2]| \geq |E_{\Gamma}[S_1]| + |E_{\Gamma}[S_2]| - |E_{\Gamma}[S_1 \cap S_2]| \tag{5}$$

$$\geq 2|S_1| - 3 + 2|S_2| - 3 - (2|S_1 \cap S_2| - 1) \tag{6}$$

$$= 2|S_1 \cup S_2| - 6 + 1. \tag{7}$$

Inequality (6) follows from Lemma 3.4 and the fact that $\Gamma[S_1 \cap S_2]$ must be (k, l) -sparse. Inequality (7) contradicts Lemma 3.2 when $l \geq 1$.

The case $l = 0$ is more elaborate. Since Γ is a planarizing gadget it must admit a planar embedding where the vertices a, c, b, d appear on the outer-face in that order, as in Figure 6. Observe that in this embedding, the set S_2 naturally partitions the set S_1 into three parts $S_a, S_1 \cap S_2, S_b$ such that $a \in S_a, b \in S_b$. Note that there cannot be any edges in the gadget with one end point in S_a and other end point in S_b . Vice versa, S_1 partitions S_2 into $S_c, S_1 \cap S_2, S_d$. We give an upper bound on the number of edges in $E_\Gamma[S_1]$ that contradicts Lemma 3.4.

For $x \in \{a, b, c, d\}$ we define

$$S'_x = S_x \cup (S_1 \cap S_2).$$

and we let $z_x \geq 0$ be an integer such that

$$|E_\Gamma[S'_x]| = 2|S'_x| - z_x.$$

Note that we can assume that there are no edges going from S_a to S_c, S_d or from S_b to S_c, S_d , otherwise (5) would be larger and thus we get a contradiction. For the same reason, we can assume $|E_\Gamma[S_1 \cap S_2]| = 2|S_1 \cap S_2|$ by (6). Thus, for the two integers $z_a, z_d \geq 0$, we have

$$\begin{aligned} |E_\Gamma[S'_a \cup S'_d]| &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_d]| - |E_\Gamma[S_1 \cap S_2]| \\ &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_d]| - 2|S_1 \cap S_2| \\ &= 2|S'_a| - z_a + 2|S'_d| - z_d - 2|S_1 \cap S_2| \\ &= 2|S'_a \cup S'_d| - (z_a + z_d). \end{aligned}$$

By Lemma 3.3, we also have $|E_\Gamma[S'_a \cup S'_d]| \leq 2|S'_a \cup S'_d| - 4$. Thus, we get

$$z_a + z_d \geq 4.$$

When we apply the same arguments to the pairs $(S'_a, S'_c), (S'_b, S'_d), (S'_b, S'_c)$ we obtain the constraints

$$\begin{aligned} z_a + z_c &\geq 4, \\ z_b + z_d &\geq 4, \\ z_b + z_c &\geq 4. \end{aligned}$$

It follows that $z_a + z_b \geq 4$ or $z_c + z_d \geq 4$. Assume $z_a + z_b \geq 4$. Since there are no edges between S_a, S_b , we have

$$\begin{aligned} |E_\Gamma[S_1]| &= |E_\Gamma[S'_a]| + |E_\Gamma[S'_b]| - |E_\Gamma[S_1 \cap S_2]| \\ &= 2|S'_a| - z_a + 2|S'_b| - z_b - 2|S_1 \cap S_2| \\ &= 2|S_1| - (z_a + z_b) \\ &\leq 2|S_1| - 4 \end{aligned}$$

But from Lemma 3.4, we must have $|E_\Gamma[S_1]| \geq 2|S_1| - 3$, which gives a contradiction. This finishes the proof of Theorem 1.1.

4 Discussion

Theorem 1.1 also holds for sparse graphs instead of tight graphs. Lemma 2.2 can be slightly modified so that edges can be replaced with a constant size graph while preserving sparsity. When we modify the proof of Theorem 1.1 for sparsity, the equality in Lemma 3.1 becomes an upper bound and the other arguments still work.

Theorem 4.1. *Planarizing gadgets do not exist for (k, l) -sparse graphs, for $k \geq 2$ and $0 \leq l \leq 2k - 1$.*

Note that Theorem 4.1 is not a generalization of Theorem 1.1, the two statements are incomparable. This is because for a (k, l) -tight graph, the graph G' where we replaced a crossing with a gadget can become (k, l) -sparse in terms of Theorem 4.1.

Our arguments do not work directly when we replace *tight* with *spanning* in Theorem 1.1 since the size of the gadget can become arbitrarily large in this case. However, note that the situation is different in this case since it is not known whether we can recognize (k, l) -spanning graphs in NC in the planar case.

Our result rules out a common approach to reduce the problem of deciding whether a given graph is (k, l) -tight or sparse to the planar case. However, this does not imply that such a reduction does not exist in general. For example, we assume that the gadget should work for all possible drawings of a given graph. One might still be able to compute a drawing of the graph in NC or Logspace such that a planarizing gadget is possible. This situation arises analogously for bipartite perfect matching and Hamiltonian cycle [GKM⁺16]. An example is illustrated in Figure 7.

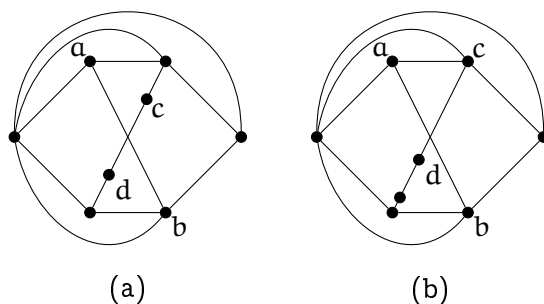


Figure 7: The same non- $(2, 3)$ -tight graph in two different drawings: (a) Adding a gadget by deleting the edges ab, cd and connecting a new vertex x to a, b, c, d makes the graph $(2, 3)$ -tight. (b) the same gadget preserves the non- $(2, 3)$ -tightness.

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A Reducing (k, l) -sparsity to max flow with multiple sources and sinks

We describe a reduction from (k, l) -sparsity to *max flow with multiple sources and sinks* (*MSMS max flow*). There are similar reductions in the literature, see e.g. [HLS97, HLS98]. Still the precise setting here needs some adaptations.

Max flow with multiple sources and sinks is defined as follows.

Definition A.1 (MSMS max flow). *We are given a tuple (G, S, T, c) , where*

- $G = (V, E)$ is a directed graph,
- $S, T \subseteq V$ are disjoint sets of sources and sinks, respectively. The vertices in S have no incoming edges and the vertices in T have no outgoing edges in G ,
- $c : E \rightarrow \mathbb{Z}_{\geq 0}$ is the capacity function.

A flow in the graph is a function $f : E \rightarrow \mathbb{R}_{\geq 0}$. For $v \in V$, the net outgoing flow from v is denoted by f_v ,

$$f_v = \sum_{(v,w) \in E} f(v,w) - \sum_{(w,v) \in E} f(w,v).$$

A flow f is feasible, if

- $\forall e \in E \quad f(e) \leq c(e)$,
- $\forall v \in V - (S \cup T) \quad f_v = 0$.

Given a feasible flow f , its value is defined as $|f| = \sum_{s \in S} f_s$. The objective of the problem is to find the maximum value of a feasible flow for (G, S, T, c) .

We show a reduction from sparseness to MSMS flows.

Lemma A.2. *Let $G = (V, E)$ be graph with $|V| = n$ and $|E| = m$, and $0 \leq l \leq 2k - 1$. Deciding whether G is (k, l) -sparse reduces to solving m many MSMS flow problems.*

Moreover, when G is planar, has bounded genus or has bounded treewidth, these flow problems will have the corresponding property too.

Proof. We construct a MSMS flow problem (H, S, T, c) as follows. Graph $H = (V_H, E_H)$ has vertices $V_H = S \cup E \cup V \cup T$, where S is a copy of E and T is a copy of V . That is, for $E = \{e_1, e_2, \dots, e_m\}$, let $S = \{s_1, s_2, \dots, s_m\}$, where $s_i \in S$ is associated with $e_i \in E$. Similarly, for $V = \{v_1, v_2, \dots, v_n\}$, let $T = \{t_1, t_2, \dots, t_n\}$, where $t_i \in T$ is associated with $v_i \in V$.

The edges of H are

$$E_H = E_1 \cup E_2 \cup E_3,$$

where

- E_1 are the S-E edges, $E_1 = \{(s_i, e_i) \mid i \in [m]\}$ with capacities $c(s_i, e_i) = 1$.

- E_2 are the E-V edges, $E_2 = \{(e, v) \mid e \in E, v \in e\}$ with capacities $c(e, v) = \alpha = m+l+1$. The value of α is chosen large enough such that the E_2 -edges can never appear in an S-T-cut of value at most $m+l$.
- E_3 are the V-T edges, $E_3 = \{(v_i, t_i) \mid i \in [n]\}$ with capacities $c(v_i, t_i) = k$.

Observe that H is constructed from G by subdividing edges or adding vertices of degree one. Hence, the construction preserves the genus, in particular planarity, and the treewidth of the input graph G .

Flow problem (H, S, T, c) serves as the base from which we define m variants that we actually use in the reduction. For each edge $e \in E$, we define essentially the same flow problem (H, S, T, c_e) , only the capacity function is changed for one edge: Let $s \in S$ be the copy of $e \in E$. Then we define c_e to be same as c except for edge (s, e) where we set

$$c_e(s, e) = l + 1.$$

The construction of (H, S, T, c_e) is shown in Figure 8.

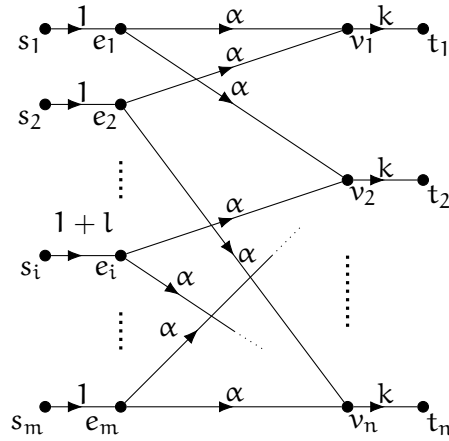


Figure 8: The flow problem (H, S, T, c_e) as described in the reduction with $e_i = e$.

We argue that G is (k, l) -sparse iff the flow problems (H, S, T, c_e) have max flow $m+l$, for all $e \in E$. By the *max-flow min-cut theorem*, we can argue via (S, T) -cuts in H . A set $C \subseteq V_H$ is a (S, T) -cut, if $S \subseteq C$ and $C \cap T = \emptyset$. The *capacity of cut* C for capacity function c_e is

$$c_e(C) = \sum_{u \in C, v \in V_H \setminus C} c_e(u, v).$$

We will show that

$$G \text{ not } (k, l)\text{-sparse} \iff \exists e \in E \exists C \subseteq V_H (S, T)\text{-cut } c_e(C) < m + l.$$

Assume first that G is not (k, l) -sparse, i.e. there is a subset $X \subseteq V$ such that $|X| \geq 2$ and $|E_G[X]| > k|X| - l$. Choose an edge $e \in E[X]$ and consider the cut $C = S \cup E_G[X] \cup X \subseteq V_H$ in H for capacity function c_e . The cut edges for C are the edges from S to $E \setminus E_G[X]$ of capacity 1 and the edges from X to T of capacity k . Hence, we have

$$\begin{aligned} c_e(C) &= m - |E_G[X]| + k|X| \\ &< k|X| + m - (k|X| - l) \\ &= m + l. \end{aligned}$$

For the backward direction, assume that for an edge $e \in E$ there is a flow problem (H, S, T, c_e) with a cut C , where $c_e(C) < m + l$. We list some properties of cut C .

- $C \neq S$, because $c_e(S) = m + l$.
- C does not cut through edges from E_2 going from E to V as they all have capacity $\alpha = m + l + 1$ and thus, the cut would be too large already from one such edge.

Therefore we have $C = S \cup F \cup X$ for some non-empty sets $F \subseteq E$ and $X \subseteq V$, such that $X' = V_G(F) \subseteq X$, where $V_G(F)$ are the endpoints of the edges in F . The latter point is because of the second item above. Note that $|X'| \geq 2$ since $|F| \geq 1$.

We have two cases depending on whether the edge e is in F or not.

Case 1. $e \in F$. Then we have

$$\begin{aligned} c_e(C) &= |E \setminus F| + k|X| \\ &= m - |F| + k|X|. \end{aligned}$$

Hence, $c_e(C) < m + l$ is equivalent to $|F| > k|X| - l$. It follows that G is not (k, l) -sparse because

$$|E_G[X']| \geq |F| > k|X| - l \geq k|X'| - l.$$

The last inequality follows because $X' \subseteq X$.

Case 2. $e \notin F$. Then we have $c_e(C) = m - |F| + l + k|X|$. Similarly as in the first case, we conclude that $|E_G[X']| > k|X'| \geq k|X'| - l$. \square

When G has bounded genus, MSMS max flow is also in NC [San18]. Given a graph G , the procedure of Sankowski [San18] first reduces the max flow problem to finding the maximum weight of an f -factor in a weighted graph G' . This in turn is reduced to the problem of finding the maximum weight perfect matching in a weighted graph G'' . Both steps of the reduction preserve planarity and the weights involved are polynomially bounded. It is easy to see that they also preserve the genus of the graph as the constructions of G' , G'' involve replacing vertices with small gadgets, and can be done for any surface embedded graph without introducing edge crossings. Therefore, the problem reduces to that of finding the

maximum weight of a perfect matching in a graph of bounded genus. This can be done in NC [AV20].

Moreover, MSMS max flow can also be solved in NC for graphs with bounded treewidth using an algorithm by Hagerup, Katajainen, Nishimura and Ragde [HKNR98].

Corollary A.3. *For $0 \leq l \leq 2k - 1$, deciding (k, l) -sparsity is in NC for graphs with bounded genus, in particular planar graphs, and for graphs with bounded treewidth.*