ON CLOSURE PROPERTIES OF GapP

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Abstract. We study the closure properties of the function classes GapP and GapP_+. We characterize the property of GapP_+ being closed under decrement and of GapP being closed under maximum, minimum, median, or division by seemingly implausible collapses among complexity classes; thereby giving evidence that these function classes don't have these closure properties.

We show a similar result concerning operations we call *bit cancellation* and *bit insertion*: Given a function $f \in \mathrm{GapP}$ and a polynomial-time computable function κ. Then we ask whether the function $f^*(x)$ that is obtained from $f(x)$ by canceling the $\kappa(x)$th bit in the binary representation of $f(x)$, or whether the function $f^+(x)$ that is obtained from $f(x)$ by inserting a bit at position $\kappa(x)$ in the binary representation of $f(x)$, is also in GapP. We give necessary conditions and a sufficient conditions for GapP being closed under bit cancellation and bit insertion, respectively.

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1. Introduction

In a fundamental paper, Valiant (1979) introduced the class #P of functions that count the number of solutions of sets in NP. Besides the algorithmical interest of determining, for example, the number of Hamiltonean cycles in a graph, the structural properties of #P have become an important subject in computational complexity theory. For example, there exists a notion of

reducibility between #P functions and the above example is known to be #P complete. A hierarchy of complexity classes, the Counting Hierarchy (Wagner 1986b), is based on #P functions and strong connections with the Polynomial Hierarchy (Stockmeyer 1977) were shown by Toda (1991).

#P has a lot of closure properties; for example, the sum or product of two #P functions is again a #P function (see, e.g., Fenner *et al.* 1991 for more closure properties). When asking for subtraction, we have to take care of the fact that counting functions are non-negative. For any $n, d > 0$, let us define the *decrement of* n *by* d as $n \ominus d = \max\{0, n - d\}$. Although this is a quite simple operation, surprisingly, it is not known whether #P is closed under decrement. The same situation we have for (integer-) division.

If one does not succeed in proving some property for some class, after a while, one might try to prove the contrary, namely that the class *does not* have this property. However, this can turn out to be a hard problem as well! Indeed, Ogiwara and Hemachandra (1993) *characterized* the property of #P being closed under decrement or division by a collapse of the Counting Hierarchy (CH = UP), thereby connecting an important open problem in computational complexity theory with some arithmetical problems concerning counting function classes. (See also Gupta 1991 for related results.) Results of this type can be read in two directions. On the one hand, since the relationship of complexity classes, like in this case, is an open question for a long time now, this explaines pretty much why it is hard to solve the above questions concerning the closure properties of #P. Furthermore, since most people conjecture that the Counting Hierarchy does not collapse to UP, this gives some evidence that #P is not closed under decrement or division. On the other hand, transforming some open problem into another one in a nontrivial way usually gives some better understanding of those problems, and, in the best case, might even give some hint on how to solve them.

Much less is known, when we weaken the question and ask for the closure of #P under decrement *for a fixed function* d, say $d = 1$; that is, "Does $f \in$ #P imply that $f \ominus 1 \in$ #P?" Torán (see Ogiwara and Hemachandra 1991) found a partial answer to this; he showed that if #P is closed under decrement by one, then NP is contained in a very small counting class (SPP), which is a generalization of the class UP (unambiguous polynomial time). But it is not known whether this collapse characterizes this property of #P, i.e., whether NP $\subseteq$ SPP implies that #P is closed under decrement by one.

The situation is similar for division: If #P is closed under division by two, then some counting classes collapse ($\oplus$P = SPP) (Ogiwara and Hemachandra 1991). However, it is not known whether the converse implication holds, that

is, whether this collapse characterizes the property of #P being closed under division by two.

Ogiwara and Hemachandra (1993) also looked for the closure of #P under the maximum, minimum, and median operator, and got the following partial result. If #P is closed under maximum, minimum, or median, then the Counting Hierarchy collapses (CH = SPP). Again, it is not known whether this collapse in turn implies the closure of #P under any of the above operations.

In this paper, we study the above questions for the classes GapP and GapP_+, where GapP is the class defined as the closure of #P under subtraction, and GapP_+ is the class of functions in GapP that are non-negative. We are able to completely characterize the property of GapP or GapP_+ of being closed under any of the above operations in terms of a seemingly implausible collapse of counting classes. More precisely, regarding the above examples, we show in Section 3 and 4 that

- GapP_+ is closed under decrement by one if and only if CH = SPP,
- GapP is closed under maximum or median if and only if CH = SPP,
- GapP is closed under division by two if and only if $\oplus$P = SPP.

We should note that the last result concerning division was already shown by Gupta (1992). But in fact, we show more general results in those sections and especially, we will improve the result of Gupta.

In Section 5, as one generalization of division and multiplication, we consider operations which we call *bit cancellation* and *bit insertion*. Let integer $n \geq 0$ in binary notation have the form $n = a_l a_{l-1} \ldots a_1 a_0$. Then $\lfloor n/2 \rfloor = a_l a_{l-1} \ldots a_1$, i.e., dividing n by two *cancels* the low order bit in the binary representation of n. More generally, let $0 \leq k \leq l$, we cancel the kth bit in the binary representation of n, and get $n^* = a_l \ldots a_{k+1} a_{k-1} \ldots a_0$. On the other hand, multiplying n by two appends one 0 to the binary representation of n. More generally, we insert one 0 at position k, and get $n^+ = a_l \ldots a_k \, 0 \, a_{k-1} \ldots a_0$. We will extend these definitions also to negative integers. We say that GapP *is closed under kth bit cancellation*, if for any GapP function f, the function f^* that is obtained from f by canceling the kth bit in the binary representation of $f(x)$ is also in GapP. We say that GapP *is closed under inserting a bit at position k*, if for any GapP function f, the function f^+ that is obtained from f by inserting a zero at position k in the binary representation of $f(x)$ is also in GapP. We give necessary conditions and sufficient conditions for the property of GapP being closed under canceling or inserting a bit at a fixed position.

2. Preliminaries

We follow the standard definitions and notations in computational complexity theory (see, e.g., Balcazar *et al.* 1988, Balcazar *et al.* 1991). We fix an alphabet to $\Sigma = \{0,1\}$; by a *string* we mean an element of Σ^*, and by a *language* we mean a subset of Σ^*. For a language L, we denote $\overline{L}$ as the complement of L, and for a class $\mathcal{C}$ of languages, co-$\mathcal{C} = \{\, \overline{L} \mid L \in \mathcal{C} \,\}$. For any string x, let $|x|$ denote the length of x, and for any set X, let $\|X\|$ denote the cardinality of X. The standard lexicographical ordering of Σ^* is used; that is, for strings $x, y \in \Sigma^*$, x is *lexicographically smaller* than y (denoted by $x < y$) if either (i) $|x| < |y|$, or (ii) $|x| = |y|$ and there exists $z \in \Sigma^*$ such that $x = z0u$ and $y = z1v$. We consider a standard one-to-one pairing function from $\Sigma^* \times \Sigma^*$ to Σ^* that is computable and invertible in polynomial time. For inputs x and y, we denote the output of the pairing function by (x, y); this notation is extended to denote every n tuple. For a function f, we simply write $f(x, y)$ instead of $f((x, y))$. A non-negative function f is *polynomially bounded*, if there exists a polynomial p such that $f(x) \leq p(|x|)$, for all $x \in \Sigma^*$.

For our computation model, we consider a standard Turing machine model. A machine is either deterministic or nondeterministic, and a deterministic machine is either an *acceptor* or a *transducer*.

- P (FP) is the class of sets (functions) computed by a deterministic polynomial-time bounded acceptor (transducer). Here, we will interpret the output of a FP function as a non-negative integer (encoded in Σ^*).
- NP is the class of sets computed by a nondeterministic polynomial-time bounded acceptor.

We also consider an *oracle machine*, i.e., a machine that can ask queries to a given oracle set. For example, $\mathrm{NP}^{\mathrm{NP}}$ is the class of sets accepted by some NP machine with an oracle set from NP. The Polynomial Hierarchy PH (Stockmeyer 1977) is defined as

- $\mathrm{PH} = \mathrm{NP} \cup \mathrm{NP}^{\mathrm{NP}} \cup \mathrm{NP}^{\mathrm{NP}^{\mathrm{NP}}} \cup \ldots$.

For counting the number of solutions of NP sets, Valiant (1979) introduced the function class #P.

- #P is the class of all functions f mapping from Σ^* to the natural numbers such that for some $A \in \mathrm{P}$ and some polynomial p, we have $f(x) = \|\{\, y \in \Sigma^{p(|x|)} \mid (x,y) \in A \,\}\|$.

#P is closed under exponential summation and polynomial products (see Fenner *et al.* 1991), but, since #P functions are non-negative, it is not closed under subtraction.

Fenner, Fortnow, and Kurtz (1991) defined the class GapP as the closure of #P under subtraction.

- $\mathrm{GapP} = \{\, f - g \mid f, g \in \#\mathrm{P} \,\}$.

Equivalently, we can define GapP as the difference of a #P function and a FP function or vice versa. We can even restrict the form of the FP function: for every $f \in \mathrm{GapP}$ and any $b \geq 2$ there is a $g \in \#\mathrm{P}$ and a polynomial p such that for all x, $f(x) = g(x) - b^{p(|x|)}$.

Even if we only subtract functions $f, g \in \#\mathrm{P}$ where $f \geq g$, it is not known whether $f - g$ is again in #P.

- $\mathrm{GapP}_+ = \{\, f \in \mathrm{GapP} \mid f \geq 0 \,\}$.

Wagner (1986) defined the Counting Function Hierarchy FCH. We give two equivalent definitions of FCH.

- $$\begin{aligned} \mathrm{FCH} &= \#\mathrm{P} \cup \#\mathrm{P}^{\#\mathrm{P}} \cup \#\mathrm{P}^{\#\mathrm{P}^{\#\mathrm{P}}} \cup \ldots \\ &= \mathrm{GapP}_+ \cup \mathrm{GapP}_+^{\mathrm{GapP}} \cup \mathrm{GapP}_+^{\mathrm{GapP}^{\mathrm{GapP}}} \cup \ldots . \end{aligned}$$

Note that we don't claim that these two definitions coincide at each level.

The following language classes PP (Gill 1977) and $\mathrm{C}_=\mathrm{P}$ (Simon 1975, Wagner 1986a) are related to the above function classes.

- PP is the class of all sets L such that there exist a #P function f and a FP function g such that for all x, we have $x \in L \iff f(x) \geq g(x)$.

We can even fix g to certain FP functions without changing the class PP. For example, we can take $g = 2^{p(|x|)}$, for some polynomial p. The same holds for the following class.

- $\mathrm{C}_=\mathrm{P}$ is the class of all sets L such that there exist a #P function f and a FP function g such that for all x, we have $x \in L \iff f(x) = g(x)$.

We can even require that $f \leq g$ (or $f \geq g$), and still get the same class $\mathrm{C_{=}P}$. Therefore, we have $\mathrm{C_{=}P} \subseteq \mathrm{PP}$. On the other hand, we have $\mathrm{PP} \subseteq \mathrm{NP}^{\mathrm{C_{=}P}}$ (Torán 1991). Alternatively, we can define PP and $\mathrm{C_{=}P}$ in terms of a GapP function instead of a #P function. Then we can even fix the FP function g in the above definitions to $g = 0$.

The Counting Hierarchy CH (Wagner 1986b) is defined as

- $\mathrm{CH} = \mathrm{PP} \cup \mathrm{PP}^{\mathrm{PP}} \cup \mathrm{PP}^{\mathrm{PP}^{\mathrm{PP}}} \cup \ldots$.

Since $\mathrm{PP}^{\mathrm{PP}} = \mathrm{PP}^{\mathrm{C_{=}P}}$ (Torán 1991), we can as well use $\mathrm{C_{=}P}$ as the oracle class over PP in the definition of CH. Clearly, $\mathrm{FCH} = \mathrm{FP}^{\mathrm{CH}}$.

For any FP function $\beta \geq 2$, the class $\mathrm{Mod}_{\beta}\mathrm{P}$ (Beigel *et al.* 1990) is defined as follows.

- $\mathrm{Mod}_{\beta}\mathrm{P}$ is the class of all sets L such that there exists a #P function f such that for all x, we have $x \in L \iff f(x) \not\equiv 0 \pmod{\beta(x)}$.

$\mathrm{Mod}_2\mathrm{P}$ is also denoted as $\oplus\mathrm{P}$ (Papadimitriou and Zachos 1983, Goldschlager and Parberry 1986).

Loosely speaking, given a base b representation of a #P function f, then a PP set can be decided by the high order bit of f and a $\mathrm{Mod}_b\mathrm{P}$ set can be decided by the low order bit of f. Toda (1991) showed that for every set L in the Polynomial Hierarchy there is a #P function f such that L can be decided by one bit of f (not necessarily the high or low order bit). This led to the definition of the MidBitP (Green *et al.* to appear) classes.

For any $b \geq 2$ and any polynomially bounded FP function κ,

- $\mathrm{MidBitP}_b(\kappa)$ is the class of all sets L such that there exists a #P function f such that for all x, we have $x \in L \iff$ the $\kappa(x)$ th bit in the b-ary representation of $f(x)$ is not 0,
- $\mathrm{MidBitP}_b$ is the union of $\mathrm{MidBitP}_b(\kappa)$ over all polynomially bounded FP functions κ.

Equivalently, for any fixed a, where $0 \leq a < b$, we can require that the $\kappa(x)$ th bit in the b-ary representation of $f(x)$ is not a. When $b = 2$, we omit the subscript, i.e., $\mathrm{MidBitP} = \mathrm{MidBitP}_2$. By the above discussion, it is clear that $\mathrm{Mod}_b\mathrm{P}$ and PP are contained in $\mathrm{MidBitP}_b$, and by the result of Toda (1991), PH is contained in $\mathrm{MidBitP}_b$, for any prime $b \geq 2$.

The class SPP (Ogiwara and Hemachandra 1991, Fenner *et al.* 1991) is defined as the GapP analogue of the class UP (Valiant 1979) (unambiguous polynomial time). In WPP and LWPP (Fenner *et al.* 1991) the strong restriction of SPP is a bit relaxed.

- SPP is the class of all sets L for which there is a function $f \in \mathrm{GapP}$ such that for all x,

$$\begin{aligned} x \in L &\implies f(x) = 1, \\ x \notin L &\implies f(x) = 0. \end{aligned}$$

- WPP is the class of all sets L for which there are functions $f \in \mathrm{GapP}$ and $g \in \mathrm{FP}$ such that for all x, we have $g(x) \neq 0$ and

$$\begin{aligned} x \in L &\implies f(x) = g(x), \\ x \notin L &\implies f(x) = 0. \end{aligned}$$

- LWPP denotes the restricted version of WPP, where the function g depends only on the length of x.

Instead of 0 and 1 in the definition of SPP, we can as well take any FP function g and require that $f(x) = g(x)$ in one case and $f(x) = g(x) + 1$ in the other case. Similar for WPP, we can take two FP function g and h and require that $f(x) = g(x)$ in one case, and $f(x) = h(x)$ in the other case.

Note that any set in WPP, where the function g in the above definition is polynomially bounded, is already in SPP. More precisely, let Gap-Few be the class of all sets L for which there exist a polynomially bounded GapP_+ function f and a P predicate Q such that for all x, we have $x \in L \iff Q(x, f(x))$ is true. Fenner, Fortnow, and Kurtz (1991) showed that Gap-Few = SPP.

Examples of sets in these classes are the Graph Automorphism problem that is in SPP and the Graph Isomorphism problem that is in LWPP (Köbler *et al.* 1992).

SPP is *low* for GapP (and for GapP_+), i.e., $\mathrm{GapP}^{\mathrm{SPP}} = \mathrm{GapP}$ (Fenner *et al.* 1991). As a consequence, SPP is contained in, and in fact low for all the above defined counting classes PP, $\mathrm{C_{=}P}$, $\mathrm{Mod}_b\mathrm{P}$ for any $b \geq 2$, WPP, LWPP, and SPP. WPP is also known to be low for PP. As a consequence, if PP would be contained in WPP or even SPP then, by an inductive argument, the whole Counting Hierarchy would collapse down to WPP or SPP, respectively. Since $\mathrm{PP} \subseteq \mathrm{SPP}^{\mathrm{C_{=}P}}$ (Torán 1991), we have $\mathrm{PP} = \mathrm{SPP} \iff \mathrm{C_{=}P} = \mathrm{SPP}$. (But it is not known whether $\mathrm{C_{=}P} = \mathrm{WPP}$ implies $\mathrm{PP} = \mathrm{WPP}$.)

It is not known whether the Counting Hierarchy collapses, but we don't expect that CH = WPP. In this sense, we take any result of the form "some assumption implies PP = WPP or even PP = SPP" as evidence that the assumption is *not* true. Similarly, regarding functions, if GapP_+ would be

contained in $\mathrm{FP}^{\mathrm{SPP}}$, then the Counting Function Hierarchy FCH would collapse to $\mathrm{FP}^{\mathrm{SPP}}$, and we can argue along the same lines as above for CH.

For any $b \geq 2$, consider the b-ary representation of a GapP function that defines some set in SPP. It is all zero, only the low order bit can be zero or one, thereby deciding membership in the SPP set. For any polynomially bounded FP function κ, we introduce the class $\mathrm{SWPP}_b(\kappa)$ (strong WPP) that is defined as SPP, but where the crucial bit is at position κ.

- $\mathrm{SWPP}_b(\kappa)$ is the class of all sets L for which there is a function $f \in \mathrm{GapP}$ such that for all x,

$$\begin{aligned} x \in L &\implies f(x) = b^{\kappa(x)}, \\ x \notin L &\implies f(x) = 0. \end{aligned}$$

- SWPP_b is the union of $\mathrm{SWPP}_b(\kappa)$ over all polynomially bounded FP functions κ.

Again, we omit the subscript when $b = 2$. For any b and κ as above, $\mathrm{SPP} \subseteq \mathrm{SWPP}_b(\kappa) \subseteq \mathrm{WPP} \subseteq \mathrm{C_{=}P} \cap \text{co-}\mathrm{C_{=}P}$, and $\mathrm{SWPP}_b(\kappa) \subseteq \mathrm{MidBitP}_b(\kappa)$. Also, for any $\kappa_1 \leq \kappa_2$ we have $\mathrm{SWPP}_b(\kappa_1) \subseteq \mathrm{SWPP}_b(\kappa_2)$.

While it is easy to see that SPP is closed under Boolean operations and $\mathrm{SWPP}_b(\kappa)$ is closed under complementation, it is not known whether $\mathrm{SWPP}_b(\kappa)$ is closed under union and intersection. Clearly, the union and intersection of two $\mathrm{SWPP}_b(\kappa)$ sets is in $\mathrm{SWPP}_b(2\kappa)$.

3. Decrement, Maximum, and Median

DEFINITION 3.1. *For integers n and d, we define* the decrement of n by d as

$$n \ominus d = \begin{cases} n - d, & \text{if } n \geq d, \\ 0, & \text{otherwise.} \end{cases}$$

Our first result states that for any fixed function $\delta \in \mathrm{FP}$, GapP_+ is not closed under decrement by δ, unless the Counting Hierarchy collapses to SPP and the Counting Function Hierarchy collapses to $\mathrm{FP}^{\mathrm{SPP}}$. Additionally, if indeed GapP_+ is not closed under decrement, then these hierarchies don't collapse to the SPP level.

THEOREM 3.2. *Let $\delta > 0$ be a polynomially bounded FP function. The following conditions are equivalent.*

(i) GapP_+ *is closed under decrement by* δ,

(ii) $\mathrm{C}_=\mathrm{P} = \mathrm{SPP}$,

(iii) $\mathrm{GapP}_+ = \mathrm{FP}^{\mathrm{SPP}}$.

PROOF. To show that (i) implies (ii), let L be a set in $\mathrm{C}_=\mathrm{P}$. Then there exists a function $f \in \mathrm{GapP}_+$ such that for all x, $x \in L \iff f(x) = 0$. Now, define g by

$$g \;=\; (\delta + 1) \cdot f \;-\; (\,(\delta + 1) \cdot f \ominus \delta\,).$$

Since $g \geq 0$ and GapP_+ is closed under decrement by δ by assumption, we have $g \in \mathrm{GapP}_+$. Furthermore, for all x, we have $g(x) = 0$, if $x \in L$, and $g(x) = \delta(x)$, otherwise. This shows $L \in \mathrm{WPP}$. But in fact, since δ is polynomially bounded, $L \in$ Gap-Few, and hence in SPP, since Gap-Few = SPP (Fenner *et al.* 1991).

To show that (ii) implies (iii), note that (ii) is equivalent with PP = SPP. Then the implication follows from the fact that $\mathrm{GapP}_+ \subseteq \mathrm{FP}^{\mathrm{PP}}$.

To show that (iii) implies (i), note that we can certainly compute the decrement of two GapP_+ functions in $\mathrm{FP}^{\mathrm{GapP}_+}$. Since $\mathrm{FP}^{\mathrm{GapP}_+} = \mathrm{FP}^{\mathrm{SPP}} = \mathrm{GapP}_+$ by assumption, we have that GapP_+ is closed under decrement. □

The simplest function we can choose is $\delta = 1$.

COROLLARY 3.3. GapP_+ *is closed under decrement by one if and only if* $\mathrm{C}_=\mathrm{P} = \mathrm{SPP}$.

If we don't have a polynomial bound on δ, then the proof of Theorem 3.2 shows that $\mathrm{C}_=\mathrm{P} = \mathrm{WPP}$. Notice that from $\mathrm{C}_=\mathrm{P} = \mathrm{WPP}$, we have $\mathrm{PP}^{\mathrm{PP}} = \mathrm{PP}^{\mathrm{C}_=\mathrm{P}} = \mathrm{PP}^{\mathrm{WPP}} = \mathrm{PP}$ since WPP is low for PP; thus, this closure property implies that the Counting Hierarchy collapses to PP.

COROLLARY 3.4. *Let $\delta > 0$ be a FP function. If* GapP_+ *is closed under decrement by* δ, *then* $\mathrm{C}_=\mathrm{P} = \mathrm{WPP}$.

Can we compute the i th smallest value of k fixed GapP functions in GapP? The following theorem says that this is not possible, unless the Counting Hierarchy collapses to SPP and the Counting Function Hierarchy collapses

to $\mathrm{FP}^{\mathrm{SPP}}$, and, if indeed GapP is not closed under this operation then these hierarchies don't collapse to the SPP level.

THEOREM 3.5. *Let $k \geq 2$ and $1 \leq i \leq k$. The following conditions are equivalent.*

(i) GapP *is closed under determining the i th smallest out of k GapP functions,*

(ii) $\mathrm{C}_{=}\mathrm{P} = \mathrm{SPP}$,

(iii) $\mathrm{GapP}_{+} = \mathrm{FP}^{\mathrm{SPP}}$.

PROOF. To show that (i) implies (ii), let L be a set in $\mathrm{C}_{=}\mathrm{P}$. Then there exists a function $f \in \mathrm{GapP}_{+}$ such that for all x, $x \in L \iff f(x) = 0$. We define GapP functions $g_1, \ldots, g_k$ and let $h(x)$ be the ith smallest value of $g_1(x), \ldots, g_k(x)$. We distinguish two cases.

Case 1: $i < k$. Define $g_1 = \ldots = g_{i-1} = -1$, $g_i = f$, and $g_{i+1} = \ldots = g_k = 1$. Then we have $h(x) = 0$, if $x \in L$, and $h(x) = 1$, if $x \notin L$. This shows $L \in \mathrm{SPP}$.

Case 2: $i = k$. Define $g_1 = \ldots = g_{k-1} = -1$ and $g_k = -f$. Then we have $h(x) = 0$, if $x \in L$, and $h(x) = -1$, if $x \notin L$. Again, we have $L \in \mathrm{SPP}$.

The proofs of the other implications are similar to the corresponding proofs of Theorem 3.2. □

Since Theorem 3.5 holds for every fixed i and $k \geq 2$, we get the same result for special operators: For $k = 2$, we get the minimum operator, when $i = 1$, and the maximum operator, when $i = 2$. For any $k \geq 2$, we get the median operator, when $i = \lfloor k/2 \rfloor$.

COROLLARY 3.6. *Let $k \geq 2$. The following conditions are equivalent.*

(i) GapP *is closed under minimum,*

(ii) GapP *is closed under maximum,*

(iii) GapP *is closed under median of k GapP functions,*

(iv) $\mathrm{C}_{=}\mathrm{P} = \mathrm{SPP}$.

4. Division

The following theorem characterizes the question of whether GapP is closed under division in terms of a collapse of some ModP classes to SPP. We conclude that it is unlikely that GapP is closed under division. This question has also been considered by Gupta (1992). But he shows the result only for division by a *polynomially bounded* FP function. Note also that Theorem 4.1 is a special case of Theorem 5.4 below. We include here anyway a proof of Theorem 4.1, because some crucial points in the proof of Theorem 5.4 can already be explained here in the simpler case.

Clearly, by division we mean integer division $\lfloor n/b \rfloor$, for integers n and $b \neq 0$.

THEOREM 4.1. *Let $\beta \geq 2$ be a FP function. GapP is closed under division by β if and only if* $\mathrm{Mod}_\beta\mathrm{P} = \mathrm{SPP}$.

PROOF. Assume that GapP is closed under division by β. Let L be a set in $\mathrm{Mod}_\beta\mathrm{P}$, i.e., there exists a function $f \in \#\mathrm{P}$ such that for all x, $x \in L \iff f(x) \not\equiv 0 \pmod{\beta(x)}$. Define the function g by

$$g = \lfloor (f + \beta - 1)/\beta \rfloor - \lfloor f/\beta \rfloor.$$

By our assumption, $g \in \mathrm{GapP}$. Let x be fixed. We can write $f(x) = \alpha \cdot \beta(x) + a$, where $\alpha \geq 0$ and $0 \leq a < \beta(x)$. Then clearly, the second term in the definition of g, $\lfloor f/\beta \rfloor = \alpha$. But the first term $\lfloor (f + \beta - 1)/\beta \rfloor$ is α only when $a = 0$, and $\alpha + 1$ otherwise. Hence, we can conclude that $x \in L \Longrightarrow g(x) = 1$ and $x \notin L \Longrightarrow g(x) = 0$. This shows that $L \in \mathrm{SPP}$.

For the reverse direction, we adapt the proof of Gupta (1992). Let F be a function in GapP. Then there exist a function $f \in \#\mathrm{P}$ and a polynomial q such that for all x, $F(x) = f(x) - \beta(x)^{q(|x|)}$. Observe that $\lfloor F(x)/\beta(x) \rfloor = \lfloor f(x)/\beta(x) \rfloor - \beta^{q(|x|)-1}$, thus it suffices to show that $\lfloor f(x)/\beta(x) \rfloor$ is in GapP.

Let $A \in \mathrm{P}$ and p be a polynomial such that $f(x) = \| \{ y \in \Sigma^{p(|x|)} \mid (x, y) \in A \} \|$. Consider the following set X. For any x and y, where $y = p(|x|)$,

$$(x, y) \in X \iff \| \{ z \in \Sigma^{p(|x|)} \mid (x, z) \in A \text{ and } z \leq y \} \| \equiv 0 \pmod{\beta(x)}.$$

For any x, there are $\lfloor f(x)/\beta(x) \rfloor$ many pairs (x, y) in X.

Obviously, X is in co-$\mathrm{Mod}_\beta\mathrm{P}$, and hence in SPP by assumption (recall that SPP is closed under complementation). Therefore, there is a function

$g \in$ GapP such that $g(x,y) = 1$, if $(x,y) \in X$, and $g(x,y) = 0$, otherwise. Now, define

$$h(x) = \sum_{y \in \Sigma^{p(|x|)}} g(x,y).$$

Obviously, h is in GapP, and we have $h(x) = \lfloor f(x)/\beta(x) \rfloor$. □

COROLLARY 4.2. *(Gupta 1992)* GapP *is closed under division by two if and only if* $\oplus$P = SPP.

Note that it is not known whether the collapse of a ModP class to SPP implies that of the Counting Hierarchy to SPP, though the converse implication holds. Thus, for the evidence that GapP is not closed under division, the collapse of a ModP class to SPP may not be as strong as that of the Counting Hierarchy.

5. Bit Cancellation and Bit Insertion

In this section, we define a generalized operation of division and multiplication which we call *bit cancellation* and *bit insertion*, respectively.

DEFINITION 5.1. *Let* n, k, *and* b *be integers, where* $k \geq 0$ *and* $b \geq 2$. *Then we define*

(1) $n^* = \lfloor n/b^{k+1} \rfloor \cdot b^k + n \bmod b^k$. *We say that* n^* *is obtained from* n *by* canceling the k th bit of n in base b, *and*

(2) $n^+ = \lfloor n/b^k \rfloor \cdot b^{k+1} + n \bmod b^k$. *We say that* n^+ *is obtained from* n *by* inserting a bit at position k in n in base b,

where by $n \bmod b^k$ *we mean* $n - \lfloor n/b^k \rfloor \cdot b^k$. *Equivalently,* $n^+ = n + (b-1) \cdot \lfloor n/b^k \rfloor \cdot b^k$.

A more intuitive explanation is the following. Let $n \geq 0$ in b-ary notation have the form $n = \alpha \cdot b^{k+1} + a \cdot b^k + \gamma$, where $\alpha \geq 0$, $0 \leq a < b$, and $0 \leq \gamma < b^{k(x)}$. Now, we *cancel the k th bit* (which is a) and get $n^* = \alpha \cdot b^k + \gamma$. When we *insert a bit at position* k, we get $n^+ = \alpha \cdot b^{k+2} + a \cdot b^{k+1} + 0 \cdot b^k + \gamma$, so in fact, we are inserting a zero at position k.

For negative n, the above explanation does not quite fit in some cases: suppose $n = -(\alpha \cdot b^{k+1} + a \cdot b^k + \gamma)$ for α, a, and γ as above. When $a > 0$ or $\gamma > 0$, then $n^* = -((\alpha+1) \cdot b^k + \gamma)$, which is different from "canceling the k th bit", i.e., $-(\alpha \cdot b^k + \gamma)$. Similarly, when $\gamma > 0$, then $n^+ = -(\alpha \cdot b^{k+2} + (a+1) \cdot b^{k+1} + \gamma)$, which is different from "inserting the k th bit", i.e., $-(\alpha \cdot b^{k+2} + a \cdot b^{k+1} + \gamma)$. Keeping this in mind, we will anyway call these operations bit cancellation and bit insertion.

Clearly, when we fix $k = 0$, then canceling the k th bit of an integer n in base b is exactly a division of n by b, and inserting a bit at position k is a multiplication of n by b. Thus, we may view these operations as generalizations of division and multiplication.

Canceling and inserting a bit from/to a GapP function can be done in polynomial time, if the function value is given. Therefore, if $\mathrm{FP}^{\mathrm{GapP}} = \mathrm{GapP}$ which, by Theorem 3.2, is equivalent with PP being SPP, then GapP is certainly closed under bit cancellation and bit insertion at any polynomial-time computable position.

PROPOSITION 5.2. *If* PP = SPP, *then* GapP *is closed under bit cancellation and bit insertion in any base.*

Our main theorems in this section are necessary conditions and sufficient conditions for the property of GapP being closed under canceling or inserting a bit at a fixed position. Although our theorems are formulated for some fixed base b in which we represent numbers, one can in fact take b as a FP function. Our main technical tool is the following lemma.

LEMMA 5.3. *Let κ and λ be polynomially bounded FP functions and $b \geq 2$. If* $\mathrm{MidBitP}_b(\kappa) \subseteq \mathrm{SWPP}_b(\lambda)$, *then for all* #P *functions f, the function $h = \lfloor f / b^{\kappa+1} \rfloor \cdot b^{2 \cdot \lambda}$ is in GapP.*

PROOF. Let $f \in \#\mathrm{P}$ and let $A \in \mathrm{P}$ and p be a polynomial such that $f(x) = \|\{\, y \in \Sigma^{p(|x|)} \mid (x, y) \in A \,\}\|$.

Consider the following sets X and Y. For any x and y, where $y = p(|x|)$,

$$(x, y) \in X \iff$$

the $\kappa(x)$ th bit of $\|\{\, z \in \Sigma^{p(|x|)} \mid (x, z) \in A \text{ and } z \leq y \,\}\|$ in base b is 0,

and

$$(x, y) \in Y \iff$$

the $\kappa(x)$ th bit of $\|\{\, z \in \Sigma^{p(|x|)} \mid (x,z) \in A \text{ and } z < y \,\}\|$ in base b is $b-1$.

For any x, there are $\lfloor f(x)/b^{\kappa(x)+1} \rfloor$ many pairs (x,y) in $X \cap Y$. To see this, observe that for every $b^{\kappa(x)+1}$ th y (in the lexicographic ordering) such that $(x,y) \in A$, we have $(x,y) \in X \cap Y$.

Obviously, X and Y are in co-MidBitP$_b(\kappa(x))$, and hence in SWPP$_b(\lambda(x))$ by assumption (recall that SWPP$_b(\lambda(x))$ is closed under complementation). Therefore, $X \cap Y \in$ SWPP$_b(2 \cdot \lambda(x))$, i.e., there is a GapP function g such that

$$g(x,y) = \begin{cases} b^{2\cdot\lambda(x)}, & \text{if } (x,y) \in X \cap Y, \\ 0, & \text{otherwise.} \end{cases}$$

Now, define $h(x) = \sum_{y \in \Sigma^{p(|x|)}} g(x,y)$. Obviously, h is in GapP and it follows from the above discussion that $h(x) = \lfloor f(x)/b^{\kappa(x)+1} \rfloor \cdot b^{2\cdot\lambda(x)}$. □

Note that we get the factor two in the exponent of $b^{2\cdot\lambda}$ simply because the intersection of two SWPP$_b(\lambda)$ sets is in SWPP$_b(2 \cdot \lambda)$. If one could show that SWPP$_b(\lambda)$ *is closed under intersection*, then we could get rid of the factor two. As a consequence, Theorem 5.4 (1) and 5.5 (1) below would become *if and only if* statements (as one can see from the proofs of the corresponding parts (2)).

Note also that Lemma 5.3 can be extended to functions f in GapP instead of #P.

THEOREM 5.4. *Let κ be polynomially bounded FP function and $b \geq 2$. Then (1) and (2) hold.*

(1) If GapP *is closed under canceling the κ th bit in base b, then* MidBitP$_b(\kappa)$ = SWPP$_b(\kappa)$.

(2) If MidBitP$_b(\kappa) \subseteq$ SWPP$_b(\lfloor \kappa/2 \rfloor)$, *then* GapP *is closed under canceling the κ th bit in base b.*

PROOF. To prove (1), assume that GapP is closed under canceling the κ th bit in base b. It suffices to show that MidBitP$_b(\kappa) \subseteq$ SWPP$_b(\kappa)$. Let L be a set in MidBitP$_b(\kappa)$, i.e., there is a function $f \in$ #P such that for all x, $x \in L$ if and only if the $\kappa(x)$ th bit of $f(x)$ in the base b representation is not zero.

For any x, we can write $f(x) = \alpha \cdot b^{\kappa(x)+1} + a \cdot b^{\kappa(x)} + \gamma$, for some $\alpha \geq 0$, $0 \leq a < b$, and $0 \leq \gamma < b^{\kappa(x)}$. We will show that there is a function $h \in$ GapP such that

$$h(x) \;=\; \begin{cases} 0, & \text{if } a = 0, \text{ i.e., } x \notin L, \\ b^{\kappa(x)}, & \text{if } a > 0, \text{ i.e., } x \in L. \end{cases} \tag{5.1}$$

From this, it clearly follows that $L \in \mathrm{SWPP}_b(\kappa)$.

Define $g = f + (b-1) \cdot b^{\kappa}$ and let f^* and g^* be the functions obtained from f and g, respectively, by canceling the κth bit. We claim that $h = g^* - f^*$ fulfills equation (5.1).

To see this, observe that

$$g \;=\; \alpha \cdot b^{\kappa+1} + (a+b-1) \cdot b^{\kappa} + \gamma \;=\; (\alpha + c) \cdot b^{\kappa+1} + a' \cdot b^{\kappa} + \gamma,$$

where

$$c = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{if } a > 0, \end{cases} \qquad \text{and} \qquad a' = \begin{cases} b-1, & \text{if } a = 0, \\ a-1, & \text{if } a > 0. \end{cases}$$

Therefore $g^* = (\alpha + c) \cdot b^{\kappa} + \gamma$, and hence, $h = g^* - f^* = c \cdot b^{\kappa}$, as we claimed above.

To prove (2), assume that $\mathrm{MidBitP}_b(\kappa) \subseteq \mathrm{SWPP}_b(\lfloor \kappa/2 \rfloor)$. Let $F \in \mathrm{GapP}$. Then there exist a function $f \in \#\mathrm{P}$ and a polynomial q such that for all x, $F(x) = f(x) - b^{q(|x|)}$. Without loss of generality, we can assume that $q(|x|) > \kappa(x)$.

Let F^* and f^* be the functions obtained from F and f, respectively, by canceling the κth bit. Observe that $F^*(x) = f^*(x) - b^{q(|x|)-1}$, i.e., it suffices to show that $f^* \in \mathrm{GapP}$.

For any x, we can write $f(x) = \alpha \cdot b^{\kappa(x)+1} + a \cdot b^{\kappa(x)} + \gamma$, for some $\alpha \geq 0$, $0 \leq a < b$, and $0 \leq \gamma < b^{\kappa(x)}$. We will show that $f^*(x) = \alpha \cdot b^{\kappa(x)} + \gamma$ is in GapP. We do this by showing that both terms (a) $\alpha \cdot b^{\kappa(x)}$ and (b) γ are GapP functions. Then, clearly, also f^* is in GapP.

(a) We apply Lemma 5.3 to f, κ, and $\lambda = \lfloor \kappa/2 \rfloor$. Then we get a GapP function h with

$$h(x) = \lfloor f(x)/b^{\kappa(x)+1} \rfloor \cdot b^{2 \cdot \lfloor \kappa(x)/2 \rfloor} = \begin{cases} \alpha \cdot b^{\kappa(x)}, & \text{if } \kappa(x) \text{ is even}, \\ \alpha \cdot b^{\kappa(x)-1}, & \text{if } \kappa(x) \text{ is odd}. \end{cases}$$

Therefore, the function $H(x)$ that is defined as $h(x)$, if $\kappa(x)$ is even, and $b \cdot h(x)$, otherwise, is the desired GapP function.

(b) To compute γ, we apply Lemma 5.3 to $f'(x) = b \cdot f(x)$ instead of $f(x)$, i.e., we shift $f(x)$ by one bit, κ, and $\lambda = \lfloor \kappa/2 \rfloor$. This gives the GapP function h' (instead of h). Now, define H' analogous to H above, then we have $H'(x) = (\alpha \cdot b + a) \cdot b^{\kappa(x)}$. Now, observe that $\gamma = f(x) - H'(x)$. □

THEOREM 5.5. *Let $\kappa > 0$ be a polynomially bounded FP function and $b \geq 2$. Then (1) and (2) hold.*

(1) *If* GapP *is closed under inserting a bit at position* κ, *then* $\mathrm{MidBitP}(\kappa - 1) \subseteq \mathrm{SWPP}(\kappa)$. [1]

(2) *If* $\mathrm{MidBitP}_b(\kappa - 1) \subseteq \mathrm{SWPP}_b(\lfloor \kappa/2 \rfloor)$, *then* GapP *is closed under inserting a bit at position* κ *in base* b.

PROOF. To prove (1), assume that GapP is closed under inserting a bit at position κ. Let L be a set in $\mathrm{MidBitP}(\kappa - 1)$, i.e., there is a function $f \in \#\mathrm{P}$ such that for all x, $x \in L$ if and only if the $(\kappa(x) - 1)$th bit in the binary representation of $f(x)$ is 1.

For any x, we can write $f(x) = \alpha \cdot 2^{\kappa(x)} + a \cdot 2^{\kappa(x)-1} + \gamma$, for some $\alpha \geq 0$, $a \in \{0, 1\}$, and $0 \leq \gamma < 2^{\kappa(x)-1}$.

Define $g = f + 2^{\kappa-1}$ and let f^+ and g^+ be the functions obtained from f and g, respectively, by inserting a bit at position κ. We have

$$\begin{array}{rcrcrcl} f^+ & = & \alpha \cdot 2^{\kappa+1} & + & a \cdot 2^{\kappa-1} & + & \gamma, \\ g^+ & = & (\alpha + a) \cdot 2^{\kappa+1} & + & (1 - a) \cdot 2^{\kappa-1} & + & \gamma. \end{array}$$

Let $h = g^+ - f^+ - 2^{\kappa-1}$. Then h is in GapP and we have $h(x) = a \cdot 2^{\kappa(x)}$. This shows that $L \in \mathrm{SWPP}(\kappa)$ via h.

It should be mentioned here that when we consider any base $b > 2$, the above proof can be extended to show that $L \in \mathrm{WPP}$. (Define $g = f + (b - 1) \cdot b^{\kappa-1}$ and $h = g^+ - f^+ - (b - 1) \cdot b^{\kappa-1}$. Then $h(x) = 0$, if $a = 0$, and $h(x) = (b - 1) \cdot b^{\kappa(x)}$ otherwise.)

To prove (2), assume that $\mathrm{MidBitP}_b(\kappa - 1) = \mathrm{SWPP}_b(\lfloor \kappa/2 \rfloor)$. Let $F \in \mathrm{GapP}$. Then there exist a function $f \in \#\mathrm{P}$ and a polynomial q such that for all x, $F(x) = f(x) - b^{q(|x|)}$. Without loss of generality, we can assume that $q(|x|) > \kappa(x)$.

Let F^+ and f^+ be the functions obtained from F and f, respectively, by inserting a bit at position κ. Observe that $F^+(x) = f^+(x) - b^{q(|x|)+1}$, i.e., it suffices to show that $f^+ \in \mathrm{GapP}$.

We apply Lemma 5.3 to f, $\kappa - 1$, and $\lambda = \lfloor \kappa/2 \rfloor$. Then we get a GapP function h with

$$h(x) = \lfloor f(x)/b^{\kappa(x)} \rfloor \cdot b^{2 \cdot \lfloor \kappa(x)/2 \rfloor} = \begin{cases} \lfloor f(x)/b^{\kappa(x)} \rfloor \cdot b^{\kappa(x)}, & \text{if } \kappa(x) \text{ is even,} \\ \lfloor f(x)/b^{\kappa(x)} \rfloor \cdot b^{\kappa(x)-1}, & \text{if } \kappa(x) \text{ is odd.} \end{cases}$$

Now, $f^+(x) = f(x) + (b - 1) \cdot h(x)$, if $\kappa(x)$ is even, and $f^+(x) = f(x) + (b - 1) \cdot h(x) \cdot b$, if $\kappa(x)$ is odd, and therefore, f^+ is in GapP. □

[1] Recall that MidBitP and SWPP are the abbreviations of $\mathrm{MidBitP}_2$ and SWPP_2, respectively.

Suppose, in Theorem 5.4 (1), we not only assume the closure of GapP under canceling the κ th bit for a fixed κ, but let κ vary over all polynomially bounded FP functions, then MitBitP collapses down to SWPP, and, since PP $\subseteq$ MidBitP, so does the whole Counting Hierarchy. The same argument applies to bit insertion.

COROLLARY 5.6. *Let $b \geq 2$. Then (1), (2), and (3) hold.*

(1) If GapP *is closed under bit cancellation in base b, then* $\mathrm{PP} = \mathrm{SWPP}_b$,

(2) if GapP *is closed under bit insertion in binary representation, then* $\mathrm{PP} = \mathrm{SWPP}$,

(3) if GapP *is closed under bit insertion in base b, then* $\mathrm{PP} = \mathrm{WPP}$.

Next, we consider the case when κ is small, where "small" means logarithmically bounded. In this range for κ, Theorems 5.4 and 5.5 become *if and only if* statements. Note that if $\kappa = O(\log n)$, then for any $b \geq 2$, $\mathrm{SWPP}_b(\kappa) = \mathrm{SPP}$ (Fenner *et al.* 1991), and for any prime b, $\mathrm{MidBitP}_b(\kappa) = \mathrm{Mod}_b\mathrm{P}$ (Beigel *et al.* 1990).

COROLLARY 5.7. *Let b be prime and $\kappa > 0$ be a logarithmically bounded FP function. The following conditions are equivalent.*

(i) GapP *is closed under canceling the κ th bit in base b,*

(ii) GapP *is closed under inserting a bit at position κ in base b,*

(iii) $\mathrm{Mod}_b\mathrm{P} = \mathrm{SPP}$.

Let us consider the case when κ is large. For a function $f \in$ GapP and a polynomial p that bounds the length of f, we call the κ th bit of f a *$O(\log n)$-highest bit of f*, if $0 \leq p - \kappa = O(\log n)$. For large κ, the corresponding MidBitP classes fall together with PP (Green *et al.* to appear). Therefore, when we apply Theorem 5.4 and 5.5 to large functions κ, we get the following.

COROLLARY 5.8. *Let $b \geq 2$. Then (1) and (2) hold.*

(1) If GapP *is closed under canceling a $O(\log n)$-highest bit in base b, then* $\mathrm{PP} = \mathrm{LWPP}$,

(2) If GapP *is closed under inserting a bit at a* $O(\log n)$*-highest position in base* b*, then* PP = LWPP.

Our last result does *not* use any FP function κ as a pointer to some bit position to be canceled. Instead, we want to cancel the highest bit of a given GapP function f *that is not* zero. In general, we cannot expect that this position is computable in polynomial time, unless the Counting Hierarchy collapses down to P.

THEOREM 5.9. *If* GapP *is closed under canceling the highest non-zero bit in binary representation, then* PP = LWPP.

PROOF. Let $L \in$ PP, i.e., there exist a function $f \in$ #P and a polynomial p such that for all x, $f(x) < 2^{p(|x|)}$, and $x \in L$ if and only if $f(x) \geq 2^{p(|x|)-1}$.

Let f^* be the function obtained from f by canceling the highest non-zero bit. By our assumption, $f^* \in$ GapP.

Define $g(x) = f(x) - f^*(x)$. Observe that $g(x) \in \{\, 2^i \mid 0 \leq i \leq p(|x|) - 1 \,\} \cup \{0\}$ and that $g(x) = 2^{p(|x|)-1} \iff x \in L$.

Now, define h by

$$h(x) = g(x) \cdot \prod_{i=0}^{p(|x|)-2} (g(x) - 2^i).$$

Note that $h(x) = 0$, if $x \notin L$, and $h(x) = 2^{p(|x|)-1} \cdot \prod_{i=0}^{p(|x|)-2}(2^{p(|x|)-1} - 2^i)$, otherwise. Since $h \in$ GapP, we have $L \in$ LWPP. □

6. Conclusions and Open Problems

We have characterized the property of GapP being closed under decrement, maximum, minimum, median, or division by seemingly implausible collapses among complexity classes. It remains open to to find characterizations for the property of GapP being closed under bit cancellation or bit insertion. One possible approach could be to show that the classes $\mathrm{SWPP}_b(k)$ are closed under complementation. Then our Theorems 5.4 and 5.5 would give such characterizations. More general, we think that it is an interesting topic for further research to investigate in which ways one can or cannot manipulate (the bits of) #P or GapP functions.

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