Reductions to Sets of Low Information Content*

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1 Overview

This paper is concerned with two basic questions about sparse sets, and a related question about sets of low instance complexity:

Question 1 With respect to what types of reductions might NP have hard or complete sparse sets?[1]

Question 2 If a set A reduces to a sparse set, does it follow that A is reducible to some sparse set that is "simple" relative to A?

Question 3 With respect to what types of reductions might NP have hard or complete sets of low instance complexity, and, relatedly, what is the structure of the class of sets with low instance complexity?

With respect to the first and third questions, intuitively one would expect that even with respect to flexible reductions NP is unlikely to have complete sets whose information content is low. With respect to the second question, one might intuitively feel that the structure imposed on a set by the fact that it reduces to a sparse set makes it plausible that we can indeed find a simple sparse set that can masquerade as the original sparse set. These two intuitions are in many ways certified by the current literature, and by the results of this paper.

The rest of this section summarizes the results of this paper and compares them with previous work.

With regard to Question 1:

- We show that if any NP-complete set bounded truth-table reduces to a set that conjunctively reduces to a sparse set, then P = NP. Our result extends the strongest previously known result, which is due to Ogiwara and Watanabe: if any NP-complete set bounded truth-table reduces to a sparse set, then P = NP [OW91]. As a consequence of our result, if any NP-complete set conjunctively reduces to a sparse set,[2] then

[1] For the reductions we will discuss, the question of sparse hard sets is equivalent to asking what type of reductions might reduce many-one complete sets for NP to some sparse set; we will often use this formulation.

[2] A conjunctive reduction from A to B means that there is a Turing machine with oracle B that accepts A, and the Turing machine's acceptance mechanism is

P = NP. The latter result has been obtained independently by Ranjan and Rohatgi [RR92].

We also show similar results for the classes $\mathrm{UP}, \mathrm{PP}, \mathrm{C_{=}P}, \mathrm{Mod}_k\mathrm{P}$, and the class of nearly near-testable sets.

- One might ask whether in the above-mentioned result of Ogiwara and Watanabe the bounded truth-table case is optimal, or whether it can be extended by making the bound on the number of queries bigger than constant, for example, by making it some function that is $\omega(\log n)$. We show that there are relativized worlds in which the boolean hierarchy does not collapse and yet there are tally NP-complete sets with respect to such reductions. This provides relativized upper bounds to the work of Ko, Orponen, Schöning, and Watanabe [KOSW86, Orp90,KOSW90], of Ogiwara and Watanabe [OW91], and of the current paper. Our result strengthens a result of Homer and Longpré [HL91], who independently of the work of this paper showed that there are relativized worlds in which there are sparse sets that are NP-complete with respect to such bounds and yet (relativized) P $\neq$ NP.

With regard to Question 2:

- In the context of recent comparisons between equivalence and reducibility to sparse sets [AHOW92,GW], it is interesting to know, for various classes of sets that reduce to sparse sets, the complexity of the easiest sparse sets to which such sets reduce. We show that any set A that disjunctively reduces (respectively, disjunctive bounded truth-table reduces, 2-truth-table reduces) to a sparse set in fact disjunctively reduces (respectively, disjunctive bounded truth-table reduces, 2-truth-table reduces) to a sparse set that is in $\mathrm{P}^{\mathrm{NP}^A}$ (respectively, $\mathrm{P}^{\mathrm{NP}^A[\log]}$, $\mathrm{P}^{\mathrm{NP}^A[\log]}$).[3] Thus, for such sets, reducing to some sparse set implies reducing to some relatively simple sparse set. The nearest previous result is one of Allender, Hemachandra,

that it accepts if and only if every string it queries is a member of B [LLS75].

[3] The [log] means that there is an $\mathcal{O}(\log n)$ bound on the number of calls made to the NP^A oracle (see [Wag90]).

Ogiwara, and Watanabe [AHOW92]: If P = NP and set A 2-truth-table reduces to a sparse set, then A truth-table reduces to some sparse set that itself truth-table reduces to A. However, A does not *two*-truth-table reduce to the particular sparse set constructed in [AHOW92]. Via census-functions, graph-coloring, and the Erdős-Rado sunflower lemma, our techniques avoid the level of explicit coding (and thus the complexity of reduction) required by previous methods.

With regard to Question 3:

- We completely characterize the sets of low instance complexity (that is, the class IC[log, poly] [KOSW86,Orp90,KOSW90]) in terms of reductions to tally sets: IC[log, poly] is exactly the class of sets that both disjunctively and conjunctively reduce to tally sets.

- We show that Orponen's result [Orp90] on 1-truth-table-complete sets for NP generalizes to disjunctive and conjunctive reductions: If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_c^p$-hard or $\leq_d^p$-hard for NP, then $A \notin \mathrm{IC}[\log, \mathrm{poly}]$.

Section 3 discusses the results associated with Question 1. Section 4 discusses the results associated with Question 2. Section 5 discusses the results associated with Question 3.

2 Notation

A set T is said to be a *tally* set if $T \subseteq 0^*$. $S^{=n}$ will denote the length n strings in S, and $S^{\leq n}$ (respectively, $S^{<n}$) will denote the strings in S of length at most (respectively, strictly less than) n. A set S is said to be *sparse* if it has at most polynomially many elements at each length: S is sparse if and only if for some polynomial p it holds that $(\forall n)[||S^{=n}|| \leq p(n)]$. We use TALLY and SPARSE to represent, respectively, the classes of tally and sparse sets. Tally and sparse sets have come to play a large role in modern complexity theory (see, e.g., the surveys [Mah86,Mah89,HOW92]).

Let $\langle \cdot, \cdot \rangle_2$ denote a (non-onto) pairing function over finite strings with the standard nice computability, and invertibility properties,

Name	Notation
many-one	$\leq_m^p$
one truth-table	$\leq_{1\text{-}tt}^p$
k truth-table	$\leq_{k\text{-}tt}^p$
k disjunctive (truth-table or Turing)	$\leq_{k\text{-}d}^p$
bounded (truth-table or Turing)	$\leq_b^p$
$f(n)$ truth-table	$\leq_{f(n)\text{-}tt}^p$
conjunctive (truth-table or Turing)	$\leq_c^p$
disjunctive (truth-table or Turing)	$\leq_d^p$
bounded conjunctive (truth-table or Turing)	$\leq_{bc}^p$
bounded disjunctive (truth-table or Turing)	$\leq_{bd}^p$
Turing	$\leq_T^p$
nondeterministic conjunctive (truth-table or Turing)	$\leq_c^{NP}$

Table 1: Polynomial-time Reductions

and such that $(\forall x, y)[|\langle x, y\rangle_2| = 2|x| + |y|]$. For every $k \geq 2$, let $\langle y_1, y_2, \ldots, y_k\rangle$ denote $\langle k, \langle y_1, \langle y_2, \langle \ldots, \langle y_{k-1}, y_k\rangle_2 \cdots\rangle_2\rangle_2\rangle_2\rangle_2$.

The reductions discussed in this paper are polynomial-bounded reductions defined by Ladner, Lynch, and Selman [LLS75]. Table 1 lists the abbreviations we will use for various types of reductions.

We will use the following notation to describe downward closures of classes under various reductions.

Notation 2.1 [BK88,AHOW92] For any reducibility $\leq_r^p$ and any class of sets $\mathcal{C}$, let $\mathrm{R}_r^p(\mathcal{C}) = \{A \mid (\exists B \in \mathcal{C})[A \leq_r^p B]\}$.

The interrelations among $\mathrm{R}_r^p(\mathrm{SPARSE})$ classes have been studied by Book and Ko [BK88], Ko [Ko89], Allender, Hemachandra, Ogiwara, and Watanabe [AHOW92], and Gavaldà and Watanabe [GW]. Figure 1 shows some of the inclusion structure among $\mathrm{R}_r^p(\mathrm{SPARSE})$ classes.

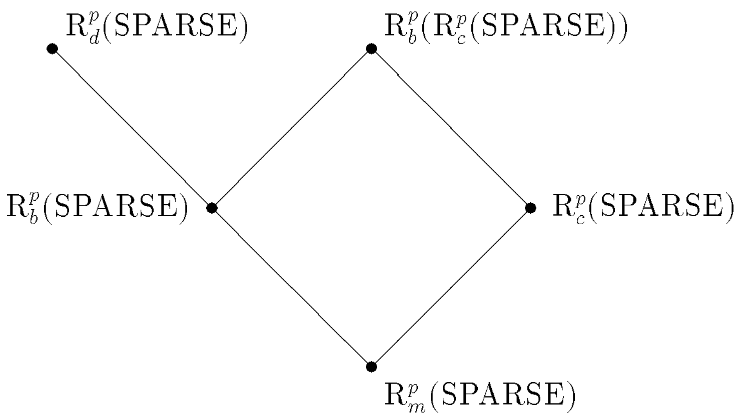

Figure 1: Inclusion structure of some reduction classes to sparse sets; all inclusions indicated are proper.

$R^p_m(\mathrm{SPARSE}) \subsetneqq R^p_b(\mathrm{SPARSE})$ and $R^p_m(\mathrm{SPARSE}) \subsetneqq R^p_c(\mathrm{SPARSE})$ are respectively from [BK88] and [Ko89]. From the result of [AHOW92] that $R^p_b(\mathrm{SPARSE}) \subseteq R^p_d(\mathrm{SPARSE})$ and the result of [GW] that $R^p_c(\mathrm{SPARSE}) \not\subseteq R^p_d(\mathrm{SPARSE})$, it follows immediately that $R^p_b(\mathrm{SPARSE}) \subsetneqq R^p_b(R^p_c(\mathrm{SPARSE}))$. From the result of [Ko89] that $R^p_b(\mathrm{SPARSE}) \not\subseteq R^p_c(\mathrm{SPARSE})$, it follows immediately that $R^p_c(\mathrm{SPARSE}) \subsetneqq R^p_b(R^p_c(\mathrm{SPARSE}))$.

It remains only to show that $R^p_b(\mathrm{SPARSE}) \subsetneqq R^p_d(\mathrm{SPARSE})$. [AHOW92] shows that $R^p_b(\mathrm{SPARSE}) \subseteq R^p_d(\mathrm{SPARSE})$, and from this it follows that coSPARSE $\subseteq R^p_d(\mathrm{SPARSE})$, which in turn implies SPARSE $\subseteq R^p_c(\mathrm{coSPARSE})$, and hence $R^p_c(\mathrm{SPARSE}) \subseteq R^p_c(R^p_c(\mathrm{coSPARSE})) = R^p_c(\mathrm{coSPARSE})$. If $R^p_b(\mathrm{SPARSE}) = R^p_d(\mathrm{SPARSE})$, then, since $R^p_b(\cdot)$ classes are closed under complement, $R^p_b(\mathrm{SPARSE}) = R^p_c(\mathrm{coSPARSE})$. The previous two sentences imply $R^p_c(\mathrm{SPARSE}) \subseteq R^p_b(\mathrm{SPARSE})$, and thus $R^p_c(\mathrm{SPARSE}) \subseteq R^p_d(\mathrm{SPARSE})$, contradicting the result of [GW].

3 Sets Reducing to Sparse Sets

The study of sparse complete sets was sparked by the conjecture of L. Berman and J. Hartmanis [BH77] that there are no sparse NP-complete sets; they were motivated to make this conjecture since if it fails then there are NP-complete sets that are not polynomial-time isomorphic (and at that time they conjectured that all NP-complete sets were polynomial-time isomorphic, though recent work has dimmed hopes on that issue [JY85,KMR89]).

The first result along the lines of their sparseness conjecture was P. Berman's proof that P = NP if some tally set is NP-complete [Ber78]. This result was quickly followed by Fortune's proof that if there is a sparse coNP-complete set, then P = NP [For79]. Finally, Mahaney obtained the striking result that P = NP if any NP-complete set many-one reduces to a sparse set [Mah82].

Although Mahaney obtained the complete collapse of the polynomial hierarchy in the case of many-one reducibility, possible collapses in the case of more flexible reducibilities have remained an active research area. For the case of Turing reductions, it is known that the existence of sparse Turing-complete sets for NP would collapse the polynomial hierarchy to $\mathrm{P}^{\mathrm{NP}[\log]}$ [Kad89], and the existence of sparse Turing-hard sets for NP would collapse the polynomial hierarchy to $\Sigma_2^p \bigcap \Pi_2^p$ [KL80]; both these results are known to be essentially optimal with respect to relativizable proof techniques [Kad89,Hel86].

As just noted, for the cases of many-one and Turing reductions the consequences of sparse NP-complete sets are well-understood. However, with respect to reductions whose strength lies between Turing and many-one reductions, the question of extending Mahaney's many-one result has proved considerably more challenging. For the case of bounded truth-table reductions, Ukkonen [Ukk83] generalized Berman's result [Ber78] by showing that if there is a tally bounded truth-table hard set for NP, then P = NP. Yesha [Yes83] generalized Fortune's result [For79] by showing that if there is a sparse bounded positive truth-table hard set for coNP, then P = NP. Yesha also (partially) generalized Mahaney's Theorem [Mah82] by showing that if there is a sparse bounded positive truth-table complete set for NP, then P = NP. These results regarding bounded truth-table reductions have been recently subsumed by Ogiwara and Watanabe ([OW91], see also [Wat88] and [JY90, footnote 9]), who successfully extended Mahaney's result and showed that if there is a sparse bounded truth-table hard set for NP, then P = NP.

For the case of conjunctive reductions, Ukkonen [Ukk83] and Yap [Yap83] generalized Fortune's result [For79] by showing that if there is a sparse conjunctive hard set for coNP, then P = NP. Yap [Yap83] also (partially) generalized Mahaney's Theorem [Mah82] by showing that if there is a sparse set that is both conjunctive and disjunctive complete for NP, then P = NP. However, in the decade since Mahaney's Theorem, it has remained an open question whether his result can be extended to the case of conjunctive reductions. Section 3.1 resolves this question.

3.1 NP, PP, and $C_=P$

We show that if there is a sparse set that is conjunctive hard for NP, then P = NP (Corollary 3.4). In fact, in Corollary 3.5 we establish that if NP $\subseteq \mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ then P = NP, thus extending the result of Ogiwara and Watanabe [OW91] (see Figure 1).

Definition 3.1 [OW91] Let A be in NP, let W be a set in P, and let q be a polynomial such that $A = \{\, x \mid (\exists\, w \in \Sigma^{q(|x|)})\,[\langle x, w\rangle \in W]\,\}$. For $x \in A$ let $w_{max}(x) = \max\{\, w \in \Sigma^{q(|x|)} \mid \langle x, w\rangle \in W\,\}$. We will say that $Left(A) = \{\, \langle x, w\rangle \mid x \in A,\ w \in \Sigma^{q(|x|)}$ and $w \leq w_{max}(x)\}$ is the *left set of A*.[4]

Theorem 3.2 If $A \in$ NP and $Left(A) \in \mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$, then A is in P.

Although the next result is a direct consequence of the above theorem, a simple direct proof for it can be found in the appendix of [AHH+92].

Theorem 3.3 If $A \in$ NP and $Left(A) \in \mathrm{R}^p_c(\mathrm{SPARSE})$, then A is in P.

[4] The left set tacitly depends on the particular witness relation chosen.

From Theorem 3.3, it immediately follows that Mahaney's Theorem generalizes to the case of conjunctive reductions. Corollary 3.4 has been obtained independently by Ranjan and Rohatgi [RR92].

Corollary 3.4 If any NP-complete set conjunctively reduces to a sparse set, then P = NP.

Indeed, $Left(A) \leq_m^p A$ for any NP-complete set A, and thus the above theorems apply to the case in which some NP-complete (or NP-hard) set reduces (via the reductions named above) to some sparse set.

Corollary 3.5 If any NP-complete set is in $\mathrm{R}_b^p(\mathrm{R}_c^p(\mathrm{SPARSE}))$, then P = NP.

In the remainder of this section we prove Theorem 3.2, give applications of the obtained results to other complexity classes, and show relativized upper bounds to the above result.

The following characterization, due to Hausdorff, of the boolean closure of certain classes of sets plays a central role in our proof of Theorem 3.2.

Theorem 3.6 [Hau14,Wec85] Let K be any class of sets closed under finite unions and intersections that includes $\emptyset$ and Σ^*. Let $BC(K)$ be the closure of K under finite union, finite intersection, and complement. Every $A \in BC(K)$ can be represented as $A = \bigcup_{i=1}^{k}(A_{2i-1} \cap \overline{A_{2i}})$, where $A_j \in K$ (for $1 \leq j \leq 2k$) and $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_{2k}$.

In order to use this characterization for sets in $R_b^p(R_c^p(\mathrm{SPARSE}))$, we need to show that $R_b^p(R_c^p(\mathrm{SPARSE})) = BC(R_c^p(\mathrm{SPARSE}))$, and that $R_c^p(\mathrm{SPARSE})$ is closed under finite unions and intersections.

Theorem 3.7 [KSW87] Let $\mathcal{K}$ be a class that contains P and is closed under many-one reductions. Then $BC(\mathcal{K}) = R_b^p(\mathcal{K})$.

By applying the recent result of Buhrman, Longpré, and Spaan (stated as Theorem 3.8 below), it remains only to prove the closure of $R_c^p(\mathrm{TALLY})$ under union.

Theorem 3.8 [BLS92] $R_c^p(\mathrm{SPARSE}) = R_c^p(\mathrm{TALLY})$.

The following lemma is straightforward and is stated without proof.

Lemma 3.9 $R_c^p(\mathrm{TALLY})$ is closed under finite unions and intersections.

Now we are ready to prove Theorem 3.2.

Proof of Theorem 3.2:
Let q be a polynomial and let P_A be a polynomial-time set such that $A = \{x \mid (\exists w \in \Sigma^{q(|x|)})[\langle x, w\rangle \in P_A]\}$. Recall that $Left(A) = \{\langle x, w\rangle \mid x \in A \ \wedge \ w \in \Sigma^{q(|x|)} \ \wedge \ w \le w_{max}\}$, where $w_{max} = \max\{w \in \Sigma^{q(|x|)} \mid \langle x, w\rangle \in P_A\}$. In the following we describe an algorithm for testing membership in A, that computes w_{max} (the lexicographically largest witness, if it exists) by a breadth-first search of the tree of prefixes of all potential witnesses. In order to do this we use the set $prefix(Left(A)) = \{\langle x, y\rangle \mid (\exists z)[\langle x, yz\rangle \in Left(A)]\}$. Each prefix y actually represents the interval of all possible extensions of y to length $q(|x|)$. It is not hard to see that $prefix(Left(A))$ is many-one equivalent to $Left(A)$ and thus $prefix(Left(A)) \in R_b^p(R_c^p(\mathrm{SPARSE})) = R_b^p(R_c^p(\mathrm{TALLY}))$.

By Theorems 3.6, 3.7, 3.8, and by Lemma 3.9, it follows that there exists a tally set T and sets $C_i \in R_c^p(T)$, $D_i \in R_d^p(T)$ such that $prefix(Left(A)) = \bigcup_{i=1}^{k}(C_i \cap D_i)$ and $C_1 \supseteq \overline{D_1} \supseteq C_2 \supseteq \overline{D_2} \ldots \supseteq C_k \supseteq \overline{D_k}$.

Let f_i be the conjunctive reduction that witnesses $C_i \in R_c^p(T)$ and g_i be the disjunctive reduction that witnesses $D_i \in R_d^p(T)$. Without loss of generality, we assume that these reductions are all computable in time bounded by a fixed polynomial p.

We first give an intuitive overview of the polynomial-time[5] algorithm recognizing A. As stated above, this algorithm performs a breadth-first search through the tree of witness prefixes for an input x. Let x be an element of A, and let $N = \{y_1, \ldots, y_t\}$ be a lexicographically ordered set of prefixes (all the same length) that includes the prefix of w_{max} of that length. We exploit some crucial properties of the Hausdorff representation $\bigcup_{i=1}^{k}(C_i \cap D_i)$ of $prefix(Left(A))$ for the design of a procedure pruning N to a polynomially size-bounded set that still includes the prefix of w_{max}.

[5] It is implicit in this section that polynomial time and polynomial size always mean polynomial in $|x|$.

Let y_m be the prefix of w_{max} in $\{y_1, \ldots, y_t\}$. Then, letting $d = 1$ and $l(0) = 1$, it holds that

$$\{\langle x, y_{l(d-1)}\rangle, \ldots, \langle x, y_m\rangle\} \subseteq C_d$$

Inductively, for $d = 1, \ldots, k$, let $r(d)$ be the largest index r such that $\{\langle x, y_{l(d-1)}\rangle, \ldots, \langle x, y_r\rangle\}$ is contained in C_d, and let $l(d)$ be the least index l such that $1 \leq l \leq r(d) + 1$ and $\{\langle x, y_l\rangle, \ldots, \langle x, y_{r(d)}\rangle\} \subseteq \overline{D_d}$. Observe that since $\{\langle x, y_{l(d-1)}\rangle, \ldots, \langle x, y_m\rangle\} \subseteq C_d$ it follows that $r(d) \geq m$. Similarly, since $\{\langle x, y_{m+1}\rangle, \ldots, \langle x, y_{r(d)}\rangle\} \subseteq \overline{D_d}$, it holds that $l(d) \leq m + 1$. We consider the following two cases separately.

1. $\langle x, y_m\rangle \in D_d$.

   Then $l(d) = m + 1$ since $y_m \notin \{y_{l(d)}, \ldots, y_{r(d)}\}$, i.e., $l(d) > m$.

2. $\langle x, y_m\rangle \notin D_d$. (This case is possible only if $d < k$.)

   In this case, $y_m \in \{y_{l(d)}, \ldots, y_{r(d)}\}$. Since $\{\langle x, y_{l(d)}\rangle, \ldots, \langle x, y_m\rangle\}$ $\subseteq prefix(Left(A))$ but $\{\langle x, y_{l(d)}\rangle, \ldots, \langle x, y_m\rangle\} \subseteq \overline{D_d}$, it follows that $\{\langle x, y_{l(d)}\rangle, \ldots, \langle x, y_m\rangle\} \subseteq C_{d+1}$, and the above analysis can be repeated.

If we could compute the prefixes $y_{l(d)}$ and $y_{r(d)}$ defined above in polynomial time, we could use the above properties in order to design a recursive procedure that collects all the prefixes $y_{l(d)-1}$ found in the recursive calls. This procedure would return a small subset of N containing y_m. Starting with $N = \{\epsilon\}$, the overall algorithm can repeatedly use such a pruning step at each level of the tree of possible witness prefixes by first expanding all the prefixes y in N to $y0$ and $y1$ (thus doubling N) and then pruning N back to a small subset. In that way, the algorithm finally computes a small subset of $\Sigma^{q(|x|)}$ that, if $x \in A$, contains w_{max}.

Although we cannot explicitly compute the required prefixes $y_{l(d)}$ and $y_{r(d)}$, instead we can compute, given $y_{l(d-1)}$, in polynomial time (polynomially size-bounded) sets $J_{right}(d)$ and $J_{left}(d)$ of prefixes such that $y_{r(d)} \in J_{right}(d)$ and $y_{l(d)} \in J_{left}(d)$. This suffices since for each prefix candidate $y \in J_{left}(d)$, the search for $y_{l(d+1)}$ can be done recursively. Since the depth of the recursion is a constant, namely k, the resulting sets $J_{left}(d)$ of candidates for $y_{l(d)}$ still have polynomially bounded cardinality.

We now describe the algorithm in detail. The algorithm calls a recursive pruning procedure PRUNE, which in turn calls two functions SEARCH-RIGHT and SEARCH-LEFT. SEARCH-RIGHT is used to search for candidates for $y_{r(d)}$ that are to the right of previously found candidates for $y_{l(d-1)}$, resulting in a polynomially size-bounded set $J_{right}(d)$ containing $y_{r(d)}$. SEARCH-LEFT is used to search to the left of the prefixes in $J_{right}(d)$, to form a polynomially size-bounded set $J_{left}(d)$ containing $y_{l(d)}$.

SEARCH-RIGHT(d, N, y_l, x)
(* returns a set $J \subseteq N = \{y_1, \ldots, y_t\}$ that includes the largest prefix $y_r \in N$ such that $\{\langle x, y_l\rangle, \ldots, \langle x, y_r\rangle\} \subseteq C_d$ *)
begin
  $J := \{y_t\}$
  for $j := 0$ **to** $p(|x|)$ **do**
    $J := J \cup \{y_h \mid$ y_{h+1} is the smallest y in N
        s.t. $y \geq y_l$ and $0^j \in f_d(\langle x, y\rangle)\}$
  end
  return J
end

Claim 1 Function SEARCH-RIGHT(d, N, y_l, x), when called with parameter $y_l = y_{l(d-1)}$, returns a set J containing $y_{r(d)}$.

Proof of Claim 1: There are two cases. If $r(d) = t$, then $y_{r(d)}$ is clearly in the returned set J. Otherwise, since $\{\langle x, y_{l(d-1)}\rangle, \ldots, \langle x, y_{r(d)}\rangle\} \subseteq C_d$ and $\langle x, y_{r(d)+1}\rangle \notin C_d$, all the queries in the sets $f_d(\langle x, y_{l(d-1)}\rangle), \ldots, f_d(\langle x, y_{r(d)}\rangle)$ are in T but at least one query 0^j in $f_d(\langle x, y_{r(d)+1}\rangle)$ is not in T. Thus $y_{r(d)+1}$ is the smallest prefix y in N such that $y \geq y_{l(d-1)}$ and $0^j \in f_d(\langle x, y\rangle)$, i.e., $y_{r(d)}$ is included in J in the j^{th} run of the for-loop. ∎

SEARCH-LEFT(d, N, y_r, x)
(* returns a set $J \subseteq N = \{y_1, \ldots, y_t\}$ that includes the smallest prefix $y_l \in N$ such that $\{\langle x, y_l\rangle, \ldots, \langle x, y_r\rangle\} \subseteq \overline{D_d}$ *)
begin
  $J := \{y_1\}$
  for $j := 0$ **to** $p(|x|)$ **do**
    $J := J \cup \{y_h \mid y_{h-1}$ is the largest y in N
      s.t. $y \leq y_r$ and $0^j \in g_d(\langle x, y\rangle)\}$
  end
  return J
end

Claim 2 Function SEARCH-LEFT(d, N, y_r, x), when called with parameter $y_r = y_{r(d)}$, returns a set J containing $y_{l(d)}$.

Proof of Claim 2: Again, there are two cases. If $l(d) = 1$, then $y_{l(d)}$ is clearly in the returned set J. Otherwise, since $\{\langle x, y_{l(d)}\rangle, \ldots, \langle x, y_{r(d)}\rangle\} \subseteq \overline{D_d}$ and $\langle x, y_{l(d)-1}\rangle \in D_d$, all the queries in the sets $g_d(\langle x, y_{l(d)}\rangle), \ldots, g_d(\langle x, y_{r(d)}\rangle)$ are outside of T but at least one query 0^j in $g_d(\langle x, y_{l(d)-1}\rangle)$ is in T. Thus $y_{l(d)-1}$ is the largest prefix y in N such that $y \leq y_{r(d)}$ and $0^j \in g_d(\langle x, y\rangle)$, i.e., $y_{l(d)}$ is included in

J in the j^{th} run of the for-loop. ∎

PRUNE(N, J'_{left}, d, x)
(* returns a subset of $N = \{y_1, \ldots, y_t\}$ that contains the prefix y_m of w_{max} if $y_m \in N \cap C_d$ and $\{y_l, \ldots, y_m\} \subseteq C_d$ for a $y_l \in J'_{left}$ with $l \leq m$ *)
begin
  if $d = k + 1$ **then return** $\emptyset$ **end**
  $J_{right} := \emptyset$
  for each $z \in J'_{left}$ **do**
    $J_{right} := J_{right} \cup$ SEARCH-RIGHT(d, N, z, x)
  end
  $J_{left} := \emptyset$
  for each $z \in J_{right}$ **do**
    $J_{left} := J_{left} \cup$ SEARCH-LEFT(d, N, z, x)
  end
  return $\{y_{l-1} \mid y_l \in J_{left}\} \cup$ PRUNE($N, J_{left}, d + 1, x$)
end

Claim 3 If $y_m \in N$, $\langle x, y_m \rangle \in C_d$, and $y_{l(d-1)} \in J'_{left}$ then function PRUNE(N, J'_{left}, d, x) returns a set I containing y_m.

Proof of Claim 3: If $y_m \in N$ and $\langle x, y_m \rangle \in C_d$ then $\langle x, y_m \rangle$ is also in the sets $\overline{D_{d-1}}, \ldots, \overline{D_1}$. By the above analysis (since case 2 always happens up to $d-1$) it follows that $\{\langle x, y_{l(d-1)} \rangle, \ldots, \langle x, y_m \rangle\} \subseteq C_d$. Since $y_{l(d-1)} \in J'_{left}$, using Claim 1, $y_{r(d)}$ is included in J_{right} by the call of SEARCH-RIGHT($d, N, y_{l(d-1)}, x$). Using Claim 2, $y_{l(d)}$ is included in J_{left} by the call of SEARCH-LEFT($d, N, y_{r(d)}, x$). Now we can prove by induction that y_m is included in the set returned by PRUNE. If $\langle x, y_m \rangle \in D_d$ (which must be true in the base case $d = k$), then $y_m = y_{l(d)-1}$ and y_m is included in the set returned by PRUNE. If $\langle x, y_m \rangle \notin D_d$ then $\langle x, y_m \rangle$ is in C_{d+1} and we can use the induction hypothesis. ∎

We complete the algorithm with a description of the main program.

```
input x
begin
N := {ε}
for i := 1 to q(|x|) do
  N := {y0 | y ∈ N} ∪ {y1 | y ∈ N}  (* expand the prefixes
                                        to length i *)
  N := PRUNE(N, {y_1}, 1, x)
end
(* N now includes w_max if x ∈ A *)
if there is a witness for x in N then accept else reject
end
end
```

In order to prove the correctness of the algorithm it suffices to observe that it follows from Claim 3 that the prefix y_m of w_{max} is included in the pruned set returned by PRUNE$(N, \{y_1\}, 1, x)$, provided that y_m is in N. Also, since the sets returned by SEARCH-RIGHT and SEARCH-LEFT are bounded in size by $p(|x|) + 2$, it follows inductively that the set J_{left} computed by PRUNE at level d is bounded in size by $(p(|x|)+2)^{2d}$. Thus, since the depth of recursion of function PRUNE is bounded by a constant, the finally returned set—being the union of all the J_{left}'s—is polynomially size-bounded, and it is easy to see that the algorithm runs in polynomial time. ∎

The Hausdorff characterization of boolean closures of classes of sets has turned out to be also useful in proving related results concerning randomized reductions and nondeterministic reductions to sparse sets (see [AKM92]).

We now briefly discuss the application of the above results to the classes UP, PP and $\mathrm{C_{=}P}$. Since for every set $A \in$ UP it holds that $Left(A)$ is in UP, it also follows that if UP is contained in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ then P $=$ UP. This strengthens the results of Watanabe [Wat91], who showed that if P $\neq$ UP then there exists a set in UP that does not many-one polynomial-time reduce to any sparse set.

Consider the set $\{\langle x, m\rangle \mid$ there are at least m satisfying assignments for $x\}$, which has properties similar to left sets and is complete for PP. Under the assumption that this set is in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$, we can use the algorithm described in the proof of Theorem 3.2 to compute in polynomial time a set of numbers that includes #SAT(x), the number of satisfying assignments of formula x. Now we can use the result of Cai and Hemachandra [CH91] and Toda (see [ABG90]) that P = PP if there is an FP function that computes on input x a set of numbers that includes #SAT(x). Alternatively, Theorem 3.10 could be proved along the same lines as Theorem 3.11.

Theorem 3.10 If PP is contained in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$, then P = PP.

We can also show that if $\mathrm{C}_=\mathrm{P}$ is contained in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ then $\mathrm{P} = \mathrm{C}_=\mathrm{P}$.

Theorem 3.11 If $\mathrm{C}_=\mathrm{P}$ is contained in $\mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ then $\mathrm{P} = \mathrm{C}_=\mathrm{P}$.

Proof of Theorem 3.11:
There exist complete sets in $\mathrm{C}_=\mathrm{P}$ that are one word-decreasing self-reducible [OL]. Balcazár has shown that every one word-decreasing self-reducible set in $\mathrm{R}^p_T(\mathrm{SPARSE})$ is in Σ^p_2 [Bal90]. So it follows from the assumption of the theorem that $\mathrm{C}_=\mathrm{P} \subseteq \Sigma^p_2$. Furthermore, since $\mathrm{coNP} \subseteq \mathrm{C}_=\mathrm{P}$, if $\mathrm{C}_=\mathrm{P} \subseteq \mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$ then also $\mathrm{NP} \subseteq \mathrm{R}^p_b(\mathrm{R}^p_c(\mathrm{SPARSE}))$, and it follows from Corollary 3.5 that $\mathrm{P} = \Sigma^p_2$. ∎

As discussed previously, Ogiwara and Watanabe [OW91] showed that if for some k NP has a k-truth-table hard sparse set then P = NP. It is natural to ask whether the result of Ogiwara and Watanabe can be extended to truth-tables that have non-constant bounds on their number of queries. We show that, even with respect to the weakened conclusion that the boolean hierarchy [CGH⁺88,CGH⁺89] collapses, the Ogiwara-Watanabe result cannot be improved to $\omega(\log n)$-bounded truth-table reductions by any relativizable proof technique. Our result strengthens the work of Homer and Longpré [HL91], who independently of the work of this paper showed that there are relativized worlds in which there are sparse sets that are NP-complete with respect to such bounds and yet $\mathrm{P} \neq \mathrm{NP}$. The strongest earlier result was the result of Kadin ([Kad89], see also [IM89,CGH⁺89]) that for every nice function $f(n) = o(\log n)$ there are relativized worlds in which the polynomial hierarchy does not collapse to $\mathrm{P}^{\mathrm{NP}[f(n)]}$ yet NP has sparse Turing complete sets. Since NP has sparse Turing complete sets if and only if NP has sparse truth-table complete sets—this can be seen either directly, via the parallel census technique discussed later in this subsection, or as a consequence of Hartmanis's sparse set that is truth-table complete for the sparse sets in NP ([Har83], only Turing completeness is stated, but Hartmanis's set is clearly truth-table complete)—Kadin's result applies equally well to the (seemingly stronger) truth-table case.

The boolean hierarchy [CGH$^+$88,CGH$^+$89] is the closure of NP under boolean operations; equivalently, it is the class of sets that can be accepted by finite amounts of hardware applied to NP predicates. Here, we define the hierarchy via one of its normal forms—namely, as the union of differences of NP sets.

Definition 3.12 [CGH$^+$88]

1. For $k \geq 1$, the *k-th* level of the boolean hierarchy is defined by: $L \in \mathrm{NP}(k)$ if and only if there exist $L_1, \ldots, L_k \in \mathrm{NP}$ such that

$$L = \begin{cases} (L_1 - L_2) \cup \ldots \cup (L_{k-2} - L_{k-1}) \cup L_k & \text{if } k \text{ is odd} \\ (L_1 - L_2) \cup \ldots \cup (L_{k-1} - L_k) & \text{if } k \text{ is even.} \end{cases}$$

2. $\mathrm{coNP}(k) = \{L \mid \overline{L} \in \mathrm{NP}(k)\}$.
3. $\mathrm{BH} = \bigcup_{k \geq 1} \mathrm{NP}(k)$, defines the *boolean hierarchy.*

Like the polynomial hierarchy, the boolean hierarchy does have downward separation; if for some k_0 it holds that $\mathrm{NP}(k_0) = \mathrm{coNP}(k_0)$, then $\mathrm{NP}(k_0) = \mathrm{BH}$ [CGH$^+$88]. In such a case, we say that the boolean hierarchy *collapses* (to level k_0).

Theorem 3.13 If f is a polynomial-time computable, nondecreasing function such that $f(n) = \omega(\log n)$, then there exist a set A and a tally set T such that BH^A does not collapse and T is $\leq^{p,A}_{f(n)\text{-}tt}$-complete for NP^A.

A detailed proof of Theorem 3.13 can be found in [AHH$^+$92]. The coding used in that proof suggests an interesting issue: given a set S of low density (for example, a sparse set), can we find its elements via *parallel* access to some NP^S set (rather than using the obvious sequential prefix-searching algorithm)? The answer, perhaps surprisingly, is that parallel access suffices. Though the tools to note this have been implicit since the important sparse set research of Hartmanis, Immerman, and Sewelson [HIS85], it is noted most clearly—in slightly different form—in a recent paper of Selman ([Sel90], see that paper for a fuller discussion of the history of this notion). The following result states that one can tighten the bound on the number of queries needed slightly beyond that found in [Sel90], and can extend the range of applicability of Selman's technique beyond the sets in NP.

Theorem 3.14 Let S be a sparse set and let d be a polynomial-time computable function such that $d(n) = n^{\mathcal{O}(1)}$ and $(\forall n)[||S^{=n}|| \leq d(n)]$. There is an $\mathrm{FP}^{\mathrm{NP}_c^S \cap \mathrm{TALLY}}_{(1+n\binom{d(n)+1}{2})\text{-}tt}$ algorithm[6] that, on input 1^n, outputs all length n strings belonging to S.[7]

A full proof of Theorem 3.14 can be found in [AHH$^+$92].

Since a conjunctive reduction is a positive reduction, if S is in NP, then $\mathrm{NP}_c^S = \mathrm{NP}$. In this case, the above result states and generalizes Selman's result that all sparse NP sets are printable via parallel access to NP. Recall that a set L is P-printable [HY84] if there is a polynomial-time computable function f such that, for every n, on input 1^n the function f outputs a list of all strings in L of length at most n; relativized P-printability is defined analogously.

Corollary 3.15 If S is a sparse set, then S is $\mathrm{P}_{\mathrm{tt}}^{\mathrm{NP}_c^S \cap \mathrm{TALLY}}$-printable. In particular, all sparse sets are $\mathrm{P}_{\mathrm{tt}}^{\mathrm{TALLY}}$-printable [Rub90], and all sparse NP sets are $\mathrm{P}_{\mathrm{tt}}^{\mathrm{NP} \cap \mathrm{TALLY}}$-printable [Sel90].

---

[6] That is, a polynomial-time machine given $1 + n\binom{d(n)+1}{2}$ parallel queries to a set in NP_c^S, the class of sets that nondeterministically conjunctively reduce [LLS75] to S.

[7] Clearly, the algorithm requires no queries for the case $d(n) = 0$, and it can be seen that there are relativized worlds in which for some set S having at most one string of each length it holds that $1 + n\binom{1+1}{2} = n + 1$ queries are actually required. We commend to the reader the open question of whether the $1 + n\binom{d(n)+1}{2}$ bound can be replaced in general by some tighter bound; we conjecture that it cannot. At issue here is the rather interesting question of the exact amount of parallel access needed to recover information about sparse sets.

3.2 Mod$_{\mathbf{k}}$P and Nearly Near-Testable Sets

Mod$_k$P [CH90,BGH90] is the class of sets L for which there is a nondeterministic polynomial-time Turing machine M such that for every x, $x \in L$ if and only if the number of accepting computation paths of M on x is not a multiple of k.

By applying a different proof technique we obtain results similar to Corollary 3.4 for the classes Mod$_k$P, $k \geq 2$.

Definition 3.16 A set L is *rotatively one word-decreasing self-reducible* if there exist a deterministic polynomial-time Turing transducer M and a polynomial p satisfying the following conditions:

1. L is a set of strings of the form $\langle x, y, i\rangle$ with $i < p(|x|)$,
2. for every x and for every y, z, there is some $d < p(|x|)$ such that for every $i < p(|x|)$, $\chi_L(\langle x, y, i\rangle) = \chi_L(\langle x, z, i \circ d\rangle)$, where $i \circ d = (i + d) \bmod p(|x|)$, and
3. for every x and y, either
   (a) $M(x, y)$ outputs $\chi_L(\langle x, y, 0\rangle) \cdots \chi_L(\langle x, y, p(|x|) - 1\rangle) \in \Sigma^{p(|x|)}$, or
   (b) $M(x, y)$ outputs $d < p(|x|)$ such that for every $i < p(|x|)$, $\chi_L(\langle x, y, i\rangle) = \chi_L(\langle x, \mathrm{pred}(y), i \circ d\rangle)$.

Theorem 3.17 Any rotatively one word-decreasing self-reducible set that conjunctively reduces to a sparse set is in P.

We prove Theorem 3.17 at the end of this section.

Since each set in Mod$_k$P, $k \geq 2$, is many-one reducible to a rotatively one word-decreasing self-reducible set in Mod$_k$P (in fact, these are essentially the strictly one word-decreasing self-reducible sets of [OL] that are complete for Mod$_k$P) we have the following corollary.

Corollary 3.18 For each $k \geq 2$: if Mod$_k$P has a sparse conjunctively-hard set then P = Mod$_k$P.

Proof of Corollary 3.18
Let L be an arbitrary set in Mod$_k$P. Let W be in P and let p be a polynomial such that for all x

$$x \in L \iff ||\{\, y \in \Sigma^{p(|x|)} \mid \langle x, y\rangle \in W \,\}|| \not\equiv 0 (\bmod k).$$

Define A to be the set of strings of the form $\langle x, y, i\rangle$ such that z is not equivalent to i modulo k, where z is the number $y' \in \Sigma^{p(|x|)}$ such that $y' \leq y$ and $\langle x, y'\rangle \in W$. Note that

- A is rotatively one word-decreasing self-reducible and
- for every x, $x \in L$ iff $\langle x, 1^{m(|x|)}, 0\rangle \in A$, and thus, L is many-one reducible to A.

So, if $\mathrm{Mod}_k\mathrm{P}$ has a sparse conjunctively-hard set S, then $A \leq_c^p S$, and so $A \in \mathrm{P}$. Thus $L \in \mathrm{P}$. ∎

For nearly near-testable sets [HH91], the class of sets that have "implicit" polynomial-time membership tests, a result similar to Theorem 3.17 holds.

Definition 3.19 [HH91] A set A is *nearly near-testable* if there exists a polynomial-time function N: $\Sigma^* \to \{\mathit{true}, \mathit{false}, \leftrightarrow, \not\leftrightarrow\}$ such that for every x one of the following holds ($x - 1$ denotes the string lexicographically preceding x):

- $N(x) = \mathit{true}$ and $x \in A$
- $N(x) = \mathit{false}$ and $x \notin A$
- $x \neq \epsilon$ and $N(x) = \leftrightarrow$ and $(x \in A \iff x - 1 \in A)$
- $x \neq \epsilon$ and $N(x) = \not\leftrightarrow$ and $(x \in A \iff x - 1 \notin A)$

Theorem 3.20 Any nearly near-testable set that conjunctively reduces to a sparse set is in P.

A full proof of Theorem 3.20 can be found in [AHH$^+$92]. Below we prove Theorem 3.17.

Proof of Theorem 3.17:

Let L be a rotatively one word-decreasing self-reducible set, as certified by machine M and polynomial p as in Definition 3.16. Suppose that L is R_c-reducible to a sparse set S via a function f. We will give a polynomial-time algorithm for L. Without loss of generality, we may assume that there exist polynomials q and r such that for every w, $f(w)$ is an encoding of a set in $\Sigma^{\leq q(|w|)}$ and for every n, $||S^{\leq q(n)}|| \leq r(n)$. Let $w_0 = \langle x_0, y_0, i_0\rangle$ be a fixed input whose membership in L we are testing. As we have fixed the input, let p, q, and r denote $p(|x_0|)$, $q(|w_0|)$, and $r(|w_0|)$, respectively. Let $I = \{0, \cdots, q-1\}$. For $a, b \in I$, let $a \circ b = (a+b) \bmod q$.

For a string y, let $\alpha(y)$ denote $\chi_L(\langle x, y, 0\rangle) \cdots \chi_L(\langle x, y, p-1\rangle)$. For a string $u \in \Sigma^p$ and $d \in I$, let $\rho(u, d)$ denote $u_{d+1} \cdots u_p u_1 \cdots u_d$. Note that, by definition, for every y, it holds that

$$(*)\ M(x_0, y) \text{ is either } \alpha(y) \text{ or } d \in I \text{ such that } \alpha(y) = \rho(\alpha(\mathrm{pred}(y)), d).$$

For a string y and $i \in I$, let $w(y, i)$ denote $\langle x_0, y, i\rangle$. An *argument sequence* is a sequence $(A_0, \cdots, A_{p-1})$ with each $A_i \subseteq \Sigma^{\leq q}$. An argument sequence $\mathcal{A} = (A_0, \cdots, A_{p-1})$ is said to be *correct* for a string y if for every $i \in I$, $w(y, i) \in L$ iff $A_i \subseteq S$. Note that if $\mathcal{A} = (A_0, \cdots, A_{p-1})$ is correct for y, then for every $i \in I$ with $||A_i|| > r$, $w(y, i) \notin L$ because $||S^{\leq q}|| \leq r$. For an argument sequence $\mathcal{A} = (A_0, \cdots, A_{p-1})$ and $d \in I$, $\mathcal{A}(d)$ denotes an argument sequence $(A_d, \cdots, A_{p-1}, A_0, \cdots, A_{d-1})$. Let $\mathcal{A} = (A_0, \cdots, A_{p-1})$ and $\mathcal{B} = (B_0, \cdots, B_{p-1})$ be given two argument sequences. We write $\mathcal{B} \subseteq_L \mathcal{A}$ to denote that for every $i \in I$ with $||A_i|| \leq r$, $B_i \subseteq A_i$. Also, $\mathcal{A} \bigcup \mathcal{B}$ denotes $(A_0 \cup B_0, \cdots, A_{p-1} \cup B_{p-1})$. For y, Φ_y denotes the argument sequence defined by $(f(w(y,0)), \cdots, f(w(y, p-1)))$. Note that Φ_y is correct for y for every y.

Claim 1 Let $\mathcal{A} = (A_0, \cdots, A_{p-1})$ and $\mathcal{B} = (B_0, \cdots, B_{p-1})$ be argument sequences that are correct for y and z, respectively. Let $d \in I$ be such that $\mathcal{B}(d) \subseteq_L \mathcal{A}$. Then, $\rho(\alpha(z), d) = \alpha(y)$.

Proof of Claim 1: Let $\mathcal{A}$, $\mathcal{B}$, y, z, and d be as in the hypothesis. Since $\mathcal{A}$ is correct for y, for every $i \in I$, $w(y, i) \in L$ iff ($||A_i|| \leq r$ and $A_i \subseteq S$). Since $\mathcal{B}(d) \subseteq_L \mathcal{A}$, for every $i \in I$ with $||A_i||$, $B_{i \circ d} \subseteq A_i$, and thus, for every $i \in I$ with $w(y, i) \in L$, $w(z, i \circ d) \in L$. Since $\mathcal{B}$ is correct for z, this implies that for every $i \in I$ with $w(y, i) \in L$, $w(z, i \circ d) \in L$. Since the number of i with $w(y, i) \in L$ is equal to the number of i with $w(z, i) \in L$, we have that for every $i \in I$ with $w(y, i) \notin L$, $w(z, i \circ d) \notin L$. Thus, $\rho(\alpha(z), d) = \alpha(y)$. ∎

Claim 2 Let $\mathcal{A} = (A_0, \cdots, A_{p-1})$ and $\mathcal{B} = (B_0, \cdots, B_{p-1})$ be argument sequences that are correct for y and z, respectively. Let $d \in I$ be such that $\mathcal{B}(d) \subseteq_L \mathcal{A}$. Then

1. if $M(x_0, z) = u$ for some $u \in \Sigma^p$, then $\rho(u, d) = \alpha(y)$, and

2. if $M(x_0, z) = e$ for some $e \in I$, then $\rho(\alpha(\mathrm{pred}(z)), e \circ d) = \alpha(y)$, and $\mathcal{A} \bigcup \Phi_y(e \circ d) = (A_0 \cup f(w(\mathrm{pred}(z), e \circ d)), \cdots, A_{p-1} \cup f(w(\mathrm{pred}(z), (p-1) \circ (e \circ d))))$ is correct for y.

Proof of Claim 2: Let $\mathcal{A}$, $\mathcal{B}$, y, z, and d be as in the hypothesis. From Claim 1, we have $\alpha(y) = \rho(\alpha(z), d)$. Suppose that $M(x_0, z) = u$ for some $u \in \Sigma^p$. By definition, $u = \alpha(z)$, and thus $\alpha(y) = \rho(u, d)$.

On the other hand, suppose that $M(x_0, z) = e$ for some $e \in I$. As discussed previously, it holds that $\alpha(z) = \rho(\alpha(\mathrm{pred}(z)), e)$. By taking $\rho(\cdot, d)$ of both sides, we have $\rho(\alpha(z), d) = \rho(\rho(\alpha(\mathrm{pred}(z)), e), d)$, and thus, $\alpha(y) = \rho(\alpha(\mathrm{pred}(z)), e \circ d)$.

By definition, $(f(w(\mathrm{pred}(z), 0)), \cdots, f(w(\mathrm{pred}(z), p-1)))$ is correct for $\mathrm{pred}(z)$. Thus, for every $i \in I$, $w(y, i) \in L$ iff $A_i \subseteq S$ iff $w(y, i \circ d \circ e) \in L$ iff $f(w(y, i \circ d \circ e)) \subseteq S$. So, $w(y, i) \in L$ iff $A_i \cup f(w(y, i \circ d \circ e)) \subseteq S$. This proves the claim. ∎

Now we define the algorithm. We operate on d, a string y, and an argument sequence $\mathcal{A} = (A_0, \cdots, A_{p-1})$. Initially, we set y to y_0, d to 0, and A_i to $f(w(y_0, i))$ for each $i \in I$, so that the following two conditions are satisfied:

(c1) $\mathcal{A}$ is correct for y_0, and

(c2) $\rho(\alpha(y), d) = \alpha(y_0)$.

The main part of the algorithm is the repetition of two steps defined below. We require that at the beginning of the first step both (c1) and (c2) hold.

First, we find $z \le y$ such that

(d1) for some $c \in I$, $\Phi_z(c) \subseteq_L \mathcal{A}$, and

(d2) either $M(x_0, z) \in \Sigma^p$ or there is no e such that $\Phi_{\mathrm{pred}(z)}(e) \subseteq_L \mathcal{A}$.

Note that, under the assumption that (c1) and (c2) hold at the beginning of this step, we have (i) every $z \le y$ satisfies at least one of these conditions, (ii) $z = y$ satisfies (d1), and (iii) $z = \epsilon$ satisfies (d2). So, by executing a simple divide-and-conquer algorithm over $[\epsilon, y]$, we can easily find z for which (d1) and (d2) are satisfied. We set c to one of the values establishing (c1).

Next, we compute $M(x_0, z)$. If this is in Σ^p, from Claim 1, $\alpha(y_0) = \rho(\alpha(z), c) = \rho(M(x_0, z), c)$. So, $w_0 = w(y_0, i_0)$ is in L iff $w(z, i_0 \circ c) \in$

L, and this is easily computed from the output of M. Hence, if this is the case, we obtain $\chi_L(w_0)$ and we accept w_0 iff it is 1. If $M(x_0, z)$ is not in Σ^p, i.e., it is in I, let e be the value and we set A_i to $A_i \cup f(w(z, i \circ c \circ e))$ for every i, and set $\mathcal{A}$ to the resulting sequence. We claim that $\mathcal{A}$ is correct for y_0. This is seen as follows. Clearly, $\Phi_{\mathrm{pred}(z)}$ is correct for $\mathrm{pred}(z)$. Since $M(x_0, z) = e$, $\Phi_{\mathrm{pred}(z)}(e)$ is correct for z. Since $\Phi_z(c) \subseteq_L \mathcal{A}$, $\Phi_{\mathrm{pred}(z)}(c \circ e)$ is correct for y_0. So $\Phi_{\mathrm{pred}(z)}(c \circ e) \bigcup \mathcal{A}$ is correct for y_0.

After executing the above two steps, we set y to $\mathrm{pred}(z)$ and d to $c \circ e$. At this point, if A_{i_0} has more than r elements, then since $A_{i_0} \subseteq S$ iff $w_0 \in L$ and it is impossible that $A_{i_0} \subseteq S$, we reject w_0 and terminate the algorithm. If A_{i_0} has at most r elements, we go back to the start of the first step and repeat these two steps.

Note that each time $\mathcal{A}$ is updated there is some $i \in I$ such that A_i gets at least one new element. So the loop is executed at most $\mathcal{O}(pr)$ times, and this is bounded by some polynomial in $|w_0|$. It is not hard to see that all the other operations can be done in time polynomial in $|w_0|$, and the algorithm correctly decides whether $w_0 \in L$. So $L \in \mathrm{P}$. ∎

4 Reductions to Simple Sparse Sets

In this section, we explore the second question mentioned in the introduction:

> If a set A reduces to a sparse set, does it follow that A is reducible to some sparse set that is "simple" relative to A?

Earlier work along these lines has been done both for the case (which is also the case of this paper) of reductions less flexible than Turing reductions [AHOW92] and for the case of Turing reductions [GW]. However, both these papers are concerned with "equivalence," and this saddles the results with weaknesses that are best illustrated by an example.

Theorem 4.1 [AHOW92] If $\mathrm{P} = \mathrm{NP}$ and set A 2-truth-table reduces to a sparse set S, then there is another sparse set $\widehat{S}$ to which A is truth-table equivalent.

The key point to notice here is that no claim is being made that A 2-truth-table reduces to $\widehat{S}$; the "equivalence" claim hides a slippage (from 2-truth-table potentially to truth-table, though in fact [AHOW92] holds the slippage to 5-truth-table) of the complexity of the reduction from A. This is not a trivial point; the slippage is crucial to the structure of earlier proofs. When reducing a set A to a sparse set via a certain fixed reducing function (we speak now not of the reduction type, such as 2-truth-table, but of the actual function that generates the queries), there will in general be many possible sparse sets to which A reduces; however, for the sparse set to be consistently defined by a reduction back to A, exactly one such set must be selected. The slippage in earlier results occurs exactly because of the cost of the disambiguating down to a single sparse set.

We now show that one can obtain results that contain no slippage at all. Very informally, to do this we avoid coding too much of the disambiguating information into the sparse set. Using this type of approach, we obtain the following results. (Note that Theorems 4.3 and 4.4 are incomparable.)

Theorem 4.2 If $A \leq_{bd}^{p} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{bd}^{p} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^{A}[\log]}$.

Theorem 4.3 If $A \leq_{2\text{-}tt}^{p} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{2\text{-}tt}^{p} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^{A}[\log]}$.

Theorem 4.4 For $k = 2$ and $k = 3$: If $A \leq_{k\text{-}d}^{p} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{k\text{-}d} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP} \oplus A}$.

Theorem 4.5 If $A \leq_{d}^{p} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{d}^{p} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^{A}}$.

Theorem 4.6 If $A \leq_{c}^{p} S$ for some sparse set S, then there is a sparse set $\widehat{S}$ such that $A \leq_{c}^{p} \widehat{S}$ and $\widehat{S} \in \mathrm{NP}^{A}$.

The rest of this section is devoted to proving Theorems 4.2 and 4.3. Detailed proofs of Theorems 4.4, 4.5, and 4.6 can be found in [AHH+92].

We introduce some notations and lemmas that will be helpful in proving Theorem 4.2. We say that $A \leq_{k\text{-}d}^{p} B$ via σ if σ is a polynomial-time function such that, for all x, $||\sigma(x)|| \leq k$ and $[x \in A \iff$

$\sigma(x) \cap B \neq \emptyset]$. A collection of distinct sets $a_1, \ldots, a_h$ is called an h-*sunflower* if the intersection $a_i \cap a_j$ is the same for every pair of distinct indices; the common part $a_i \cap a_j$ is called the center of the sunflower. A collection W of sets is called h-*compact* if there are no subcollections of W that are $(h+1)$-sunflowers. The following combinatorial lemma about sunflowers, due to Erdős and Rado [ER60] (see also [BS90]), will be used extensively.

Lemma 4.7 [ER60] If W is an h-compact collection of sets, each of cardinality at most k, then there are at most $h^k k!$ sets in W.

The proof of Theorem 4.2 is obtained from the following technical lemma.

Lemma 4.8 Let $k \geq 1$. Suppose that $A \leq^p_{k\text{-}d} S$ via σ and B, D are two sets such that:

1. S is a sparse set,
2. $D \in \mathrm{NP}^{A \oplus B}$, $S \cap D = \emptyset$, and
3. σ satisfies the following "honesty" condition: a polynomial-time function r exists such that, for all z and y, $z \in \sigma(y) \Rightarrow |y| = r(0^{|z|})$, and for some polynomial p, $(\forall n)\,[r(0^n) \leq p(n)]$.

Then there is a sparse set $\widehat{S}$ such that $A \leq^p_{k\text{-}d} \widehat{S}$ via σ, $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^{A\oplus B}[\log]}$, and $\widehat{S} \cap D = \emptyset$.

Proof of Lemma 4.8:
The idea can be expressed intuitively as follows. If many strings in A query the same strings in a set c of cardinality less than k, then a simple sparse set S' can be found by induction. Other queries can be proved by Lemma 4.7 to be covered by a sparse set S'' whose simplicity, relative to A, is shown by an algorithm for checking h-compactness.

The proof is by induction on k. If $k = 1$, define $\widehat{S} = \{z \mid (\exists y)[|y| = r(0^{|z|})$ and $y \in A$ and $z \in \sigma(y)]\}$. It is easy to verify that $\widehat{S}$ satisfies the thesis. Indeed, in this case it even holds that $\widehat{S} \in \mathrm{NP}^A$.

Let $k > 1$, and suppose that $A \leq^p_{k\text{-}d} S$ via σ and B, D are two sets that satisfy conditions (1)-(3). Since σ is computable in polynomial time and S is a sparse set, there exists a polynomial p such that, for all y and z, $z \in \sigma(y) \Rightarrow |z| \leq p(|y|)$, and, for all n, $||S^{\leq n}|| \leq p(n)$.

Let q be a polynomial such that $q(n) > p(p(n))$. The crucial fact that allows us to apply the inductive hypothesis is the following claim, which follows from the sparseness bound and is stated without proof.

Claim 1 If $y_1, \ldots, y_h \in A$ are such that $|y_1| = \cdots = |y_h|$, $h = q(|y_1|)$, and the collection of sets $\sigma(y_1), \ldots, \sigma(y_h)$ is an h-sunflower whose center is c, then $c \cap S \neq \emptyset$.

Observe that, in the case described in Claim 1, the cardinality of the center c certainly is strictly less than k. For each $m = 1, \ldots, k-1$, define A_m and σ_m as follows: $A_m = \{\langle y_1, \ldots, y_h\rangle \mid y_1, \ldots, y_h \in A$ and $|y_1| = \cdots = |y_h|$ and $h = q(|y_1|)$ and $\{\sigma(y_1), \ldots, \sigma(y_h)\}$ is an h-sunflower whose center has cardinality m $\}$, and

$$\sigma_m(y) = \begin{cases} c & \text{if } y = \langle y_1, \ldots, y_h\rangle \text{ and } |y_1| = \cdots = |y_h| \text{ and} \\ & h = q(|y_1|) \text{ and } \{\sigma(y_1), \ldots, \sigma(y_h)\} \text{ is an } h\text{-sunflower} \\ & \text{whose center } c \text{ has cardinality } m \\ \emptyset & \text{otherwise.} \end{cases}$$

Now, we prove that for all m, $1 \leq m \leq k-1$, it holds that $A_m \leq^p_{m\text{-}d} S$ via σ_m. Take m such that $1 \leq m \leq k-1$. Suppose that $y \in A_m$. Then there are $y_1, \ldots, y_h$ such that $y = \langle y_1, \ldots, y_h\rangle$, $h = q(|y_1|)$, $y_1, \ldots, y_h \in A$, $|y_1| = \cdots = |y_h|$, and $\{\sigma(y_1), \ldots, \sigma(y_h)\}$ is an h-sunflower whose center c has cardinality m; thus $\sigma_m(y) = c$, and from Claim 1 it holds that $c \cap S \neq \emptyset$, and so $\sigma_m(y) \cap S \neq \emptyset$. If $y = \langle y_1, \ldots, y_h\rangle \notin A_m$, then we have two cases: if there is a string y_i that does not belong to A, then $\sigma(y_i) \cap S = \emptyset$ and thus $\sigma_m(y) \cap S = \emptyset$ (since $\sigma_m(y) \subseteq \sigma(y_i)$); on the other hand, if every y_i belongs to A, then, since $y \notin A_m$, it must be the case that $\sigma_m(y) = \emptyset$.

Let $OUT = \{z \mid (\exists y)[|y| = r(0^{|z|}) \text{ and } y \notin A \text{ and } z \in \sigma(y)]\}$, i.e., the set of strings that are "provably" out of S. Let $D' = D \cup OUT$. It is easy to see that, for all m, $1 \leq m \leq k-1$, it holds that A_m, S, σ_m, $A \oplus B$, and D' satisfy conditions (1)-(3) (with $A_m \to A$, $S \to S$, $\sigma_m \to \sigma$, $A \oplus B \to B$, $D' \to D$), so we can apply the inductive hypothesis, which ensures, for every $1 \leq m \leq k-1$, the existence of a sparse set S_m such that $A_m \leq^p_{m\text{-}d} S_m$ via σ_m, $S_m \in \mathrm{P}^{\mathrm{NP}^{A_m \oplus (A \oplus B)}[\log]}$, and $S_m \cap D' = \emptyset$. Define $S' = S_1 \cup \cdots \cup S_{k-1}$; since $A_m \in \mathrm{P}^A$, it holds that $S' \in \mathrm{P}^{\mathrm{NP}^{A \oplus B}[\log]}$. Furthermore, if $y \notin A$ then $\sigma(y) \cap S' = \emptyset$ (since $S' \cap D' = \emptyset$). Unfortunately, if $y \in A$ we cannot prove that $\sigma(y) \cap S' \neq \emptyset$, thus we have to add other elements to S'. Consider the collection of sets that are not "covered" by S' : $V = \{a \mid (\exists y)[y \in A$ and $\sigma(y) = a]$ and $a \cap S' = \emptyset\}$. This collection of sets can be subdivided into subcollections: $V_n = \{a \mid a \in H_n$ and $a \cap S' = \emptyset\}$, where $H_n = \{a \mid (\exists y)[y \in A$ and $|y| = n$ and $\sigma(y) = a]\}$. Clearly $V = \bigcup_n V_n$. The collection V_n has the following important property.

Claim 2 If $a \in V_n$ then there is no collection E such that $E \subseteq H_n$, $a \in E$, and E is a $q(n)$-sunflower.

Proof of Claim 2: Let $a \in V_n$. Suppose that there exists a collection $E \subseteq H_n$ with $a \in E$, and E is a $q(n)$-sunflower. Then there exist $y_1, \ldots, y_h$ with $h = q(n)$, such that $y_1, \ldots y_h \in A$, $|y_1| = \cdots = |y_h| = n$, $\sigma(y_1) = a$, and $\{\sigma(y_1), \ldots, \sigma(y_h)\} = E$. Let c be the center of E, and $m = ||c|| < k$. Thus it holds that $\langle y_1, \ldots, y_h \rangle \in A_m$ and $\sigma_m(\langle y_1, \ldots, y_h \rangle) = c$, and, since $A_m \leq^p_{m\text{-}d} S_m$, it follows that $c \cap S_m \neq \emptyset$. Furthermore, $c \subseteq \sigma(y_1) = a$, and thus $a \cap S_m \neq \emptyset$, which contradicts the assumption that $a \in V_n$. ∎

Claim 2 suggests consideration of the following collection: $W_n = \{a \mid a \in H_n$ and there is no collection E such that $E \subseteq H_n$, $a \in E$, and E is a $q(n)$-sunflower $\}$. Define $S'' = \{z \mid (\exists a)[a \in W_{r(0^{|z|})}$ and $z \in a]\} - D'$, and $\widehat{S} = S' \cup S''$. From the definition of W_n it is clear that W_n is $(q(n)-1)$-compact, thus by Lemma 4.7 S'' is a sparse set. Furthermore, it is not hard to verify that $A \leq^p_{k\text{-}d} \widehat{S}$ via σ. It remains to show that $S'' \in \mathrm{P}^{\mathrm{NP}^{A \oplus B}[\log]}$. A naïve algorithm, based directly on the definitions of S'' and W_n, yields only $S'' \in \Sigma_2^{p, A \oplus B}$. In order to accomplish our goal we need the following algorithm, which, given a collection of sets T and an integer h, "approximately" checks whether or not T is h-compact. We assume a total ordering of all finite sets of strings.

APPROX(T,h)
begin
  for each subset c of some set $a \in T$ **do**
    $I := \emptyset$
    $m := 0$
    while $m = 0$ **do**
      find (if such exists) the minimum (with respect to
      the total ordering of finite sets) set $a \in T$ such that:
      (1) $a \notin I$, (2) $c \subseteq a$, and (3) $(\forall b \in I)[b \cap a = c]$
      if such a exists **then** $I := I \cup \{a\}$
      else $m := ||I||$
      end
    end (*while*)
    if $m > h$ **then** *reject* **end**
  end (*for*)
  accept
end

The above algorithm has the following properties.

1. If the cardinalities of the sets of the input collection T are bounded by a constant then APPROX runs in polynomial time.
2. If $(\forall a \in T)[||a|| \leq k]$ and APPROX(T,h) accepts then T is kh-compact.
3. If APPROX(T,h) rejects then T is not h-compact.
4. If APPROX(T,h) accepts and APPROX($T \cup \{a\}$,h) rejects then there is a collection $E \subseteq T \cup \{a\}$ such that $a \in E$ and E is a $(h+1)$-sunflower.

Properties (1), (3), and (4) are easy to prove. In order to prove property (2) we need the following.

Claim 3 Let E and F be two collections of sets with each set of cardinality at most k. If E is a m-sunflower and F is a ℓ-sunflower whose center is the same of that of E, then there exist at least $m - \ell k$ sets a in E such that $F \cup \{a\}$ is a $(\ell + 1)$-sunflower.

Proof of Claim 3: Let $E = \{a_1, \ldots, a_m\}$, $F = \{b_1, \ldots, b_\ell\}$, and let c be the common center of E and F. Clearly, if $F \subseteq E$ then the assertion is true. Hence, in the following we suppose that $F \not\subseteq E$.

The proof is by induction on ℓ. Let $\ell = 1$, define, for each $i = 1, \dots, m$, $d_i = (a_i \cap b_1) - c$. Since $\{a_1, \dots, a_m\}$ is a m-sunflower whose center is c, it holds that, for all distinct i, j, $d_i \cap d_j = \emptyset$. Furthermore, for each $i = 1, \dots, m$, $d_i \subseteq b_1$ and $||b_1|| \leq k$, thus there are at least $m - k$ sets d_i such that $d_i = \emptyset$. It follows that there are at least $m - k$ sets a in E such that $a \neq b_1$ and $a \cap b_1 = c$, that is, $\{b_1, a\}$ is a 2-sunflower.

Let $\ell > 1$, since $F \not\subseteq E$, we can assume that $b_\ell \notin E$. Applying the inductive hypothesis to $\{b_1, \dots, b_{\ell-1}\}$ we obtain that there are at least $m - (\ell - 1)k$ sets a in E such that $\{b_1, \dots, b_{\ell-1}, a\}$ is a ℓ-sunflower. Without loss of generality, we can assume that, for each $i = 1, \dots, n$ with $n = m - (\ell - 1)k$, $\{b_1, \dots, b_{\ell-1}, a_i\}$ is a ℓ-sunflower. Define, for each $i = 1, \dots, n$, $d_i = (a_i \cap b_\ell) - c$. Since $\{a_1, \dots, a_n\}$ is a n-sunflower whose center is c, it must be the case that, for all distinct i, j, $d_i \cap d_j = \emptyset$. Furthermore, for each $i = 1, \dots, n$, $d_i \subseteq b_\ell$ and $||b_\ell|| \leq k$. Thus, there are no less than $n - k$ sets d_i such that $d_i = \emptyset$. It follows that there are at least $m - \ell k$ sets a in E such that $F \cup \{a\}$ is a $(\ell + 1)$-sunflower. ∎

Suppose that T is not kh-compact. This means that a subcollection E of T exists that is a $(kh + 1)$-sunflower. Let c be the center of E. Claim 3 implies that the while-loop of APPROX(T,h), relative to c, makes at least $h+1$ iterations, and thus APPROX(T,h) rejects. This contradicts the assumption that APPROX(T,h) accepts.

A useful consequence of property (4) is the following.

Claim 4 If $T \subseteq H_n$ and APPROX(T,$q(n)$) accepts then APPROX($T \cup W_n$, $q(n)$) accepts.

Proof of Claim 4: By induction on the cardinality of W_n, using property (4) of APPROX. ∎

Let $m_n = \max\{||T|| \mid T \subseteq H_n$ and APPROX(T,$q(n)$) accepts$\}$. From Claim 4 we have that if $T \subseteq H_n$, APPROX(T,$q(n)$) accepts and $||T|| = m_n$ (T is maximal), then $W_n \subseteq T$. Thus, if we knew m_n and $||W_n||$, then given any string z with $r(0^{|z|}) = n$ we could check via one query to a suitable NP^A oracle whether or not there is a set $a \in W_n$ such that $z \in a$. In fact, an NP^A machine can guess a collection $T \subseteq H_n$ and verify that $||T|| = m_n$ and that APPROX(T,$q(n)$) accepts (observe that from property (2) of APPROX and Lemma 4.7 there is a polynomial that bounds m_n), subsequently (let $h = m_n - ||W_n||$) the NP^A machine guesses sets $a_1, \ldots, a_h$ and collections $E_1, \ldots, E_h$ such that $E_i \subseteq H_n$, $a_i \in E_i \cap T$, and $a_i \neq a_j$, and verifies that for every i it holds that E_i is a $q(n)$-sunflower; at this point, the computation accepts if and only if there is a set $a \in T - \{a_1, \ldots, a_h\}$ such that $z \in a$.

It is not hard to see that m_n and subsequently $||W_n||$ can be computed in polynomial time via $\mathcal{O}(\log n)$ queries to a suitable NP^A oracle. ∎

Proof of Theorem 4.2:
Let $A \leq^p_{k\text{-}d} S$ via σ, and let S be a sparse set. Define $S' = \{\langle 0^l, x\rangle_2 \,|\, x \in S \text{ and } l \geq 0\}$, and, for each x, $\sigma'(x) = \{\langle 0^{|x|p(|x|)}, y\rangle_2 \,|\, y \in \sigma(x)\}$, where p is a polynomial such that, for all y and z, $z \in \sigma(y) \Rightarrow |z| \leq p(|y|)$. It is easy to verify that $A \leq^p_{k\text{-}d} S'$ via σ', and that A, S', and σ' satisfy conditions (1)-(3) of Lemma 4.8 (with $B = D = \emptyset$). Thus, applying Lemma 4.8, we obtain a sparse set $\widehat{S}$ such that $A \leq^p_{k\text{-}d} \widehat{S}$ and $\widehat{S} \in \mathrm{P}^{\mathrm{NP}^A[\log]}$. ∎

We now turn to the proof of Theorem 4.3. Recall, as we discussed earlier, that when reducing a set A to a sparse set via a certain fixed reducing function, there will usually be many possible sparse sets to which A reduces. For the sparse set to be consistently defined by a reduction back to A, exactly one such set must be selected. The following proof tries to eliminate this ambiguity as follows. Via a binary search for census information, the machine to accept a sparse set (via access to the original set) obtains information about it. The search is "promise"-like (see, e.g., [Sel88]) in that it works only because the census is found to be sparse in the previous step. Thus, the machine for the sparse set has obtained census information about certain subsets of the strings in and out of itself; using this, it reduces (taking the case of Proposition 4.18 as an example) all remaining ambiguity to a feasible graph coloring problem. Crucially, all inputs (of the same connected component) will obtain the same coloring problem, and this will cause the sparse set to be consistently and correctly defined.

Proof of Theorem 4.3:
Without loss of generality, it is assumed that exactly two queries are asked in the 2-truth-table reductions. Given an input y, a 2-truth-table reduction generates a truth-table as well as a pair of

queried strings; the acceptance of the input is decided by looking up the truth-table using the result of queries as an index. There are sixteen different truth-tables that can be generated; they are shown in Figures 2–7. (In the tables, entry 1 means acceptance, and 0 means rejection.) Let $\tau(y)$ denote the truth-table generated on the input y; and, let $\mathcal{T}$ be the set of sixteen truth-tables of arity two. For simplicity we will denote each member of $\mathcal{T}$ by its corresponding table number in Figures 2–7. For each $t \in \mathcal{T}$, let $B_t = \{y \in \Sigma^* \mid \tau(y) = t\}$. The following proposition is clear.

Proposition 4.9 Let $\tau(\cdot)$ be a polynomial-time computable function that on an input y, generates a truth-table in $\mathcal{T}$. Then the following holds.
(i) $(\forall t \in \mathcal{T})\ B_t \in \mathrm{P}$,
(ii) $(\forall t, t' \in \mathcal{T})\ [t \neq t' \Longrightarrow B_t \cap B_{t'} = \emptyset]$,
(iii) $\bigcup_{t \in \mathcal{T}} B_t = \Sigma^*$.

This proposition says that given a 2-truth-table reduction, the input string domain is completely covered by disjoint polynomial-time sets each of which is associated with a corresponding truth-table. Henceforth, we will call $\{B_t \mid t \in \mathcal{T}\}$ the decomposition of the input string domain associated with the given 2-truth-table reduction.

In order to be able to discuss separately each of the truth-tables that may be used in a 2-truth-table reduction, we introduce the following definition.

Definition 4.10 Fixed truth-table reductions [Wec85] A truth-table reduction is called a fixed truth-table reduction if it uses the same truth-table for all its inputs. We denote such a reduction with $\leq_t^p$, where t denotes a truth-table: $A \leq_t^p B$ means $A \leq_{tt}^p B$ with the fixed truth-table t. We also denote such a reduction with $\leq_{ftt}^p$: i.e., $A \leq_{ftt}^p B$ *with* t means $A \leq_{tt}^p B$ with the fixed truth-table t. Similarly, $A \leq_{k\text{-}ftt}^p B$ *with* t means $A \leq_{k\text{-}tt}^p B$ with the fixed truth-table t.

The next two propositions, together with Proposition 4.9 enable us to find a corresponding sparse set individually for each fixed truth-table reduction, and combine the results together to form a full sparse set. The proofs are straightforward and we leave them as a exercise to the reader. Note that in Proposition 4.11, the *truth-table* 2 is given a special treatment since the only set that 2-truth-table reduces via the *truth-table* 2 is Σ^*.

Proposition 4.11 Let A be a set that 2-truth-table reduces to a sparse set. Let $\{B_t \mid t \in \mathcal{T}\}$ be the decomposition of the input string domain associated with the 2-truth-table reduction. Then the following holds.
(i) $(\forall t \in \mathcal{T})\ [t \neq 2 \implies (\exists S_t \in \text{SPARSE})\ B_t \cap A \leq_t^p S_t]$,
(ii) $t = 2 \implies (\exists S_t \in \text{SPARSE} \cap \text{P})\ B_t \cap A \leq_{2\text{-}tt}^p S_t$.

Proposition 4.12 Let $\mathcal{C}$ be a complexity class such that for every oracle D (i) $\mathcal{C}^D$ is closed under $\leq_m^p$, (ii) $\mathcal{C}^D$ has a $\leq_m^p$-complete set, and (iii) for any $B \in \text{P}$, $\mathcal{C}^{B \cap D} \subseteq \mathcal{C}^D$. Let A be a set that 2-truth-table reduces to a sparse set. Let $\{B_t \mid t \in \mathcal{T}\}$ be the decomposition of the input string domain associated with the 2-truth-table reduction. Suppose that for all $t \in \mathcal{T}$, there exists a sparse $S_t \in \mathcal{C}^{B_t \cap A}$ such that $B_t \cap A \leq_{2\text{-}tt}^p S_t$. Then there exists a sparse set $\widehat{S} \in \mathcal{C}^A$ such that $A \leq_{2\text{-}tt}^p \widehat{S}$.

Clearly, for every oracle D, $\text{P}^{\text{NP}^D[\log]}$ qualifies as the complexity class $\mathcal{C}^D$ in Proposition 4.12. To establish the theorem, it remains to show that Proposition 4.11 implies that for all $t \in \mathcal{T}$, there exists a sparse $S_t \in \text{P}^{\text{NP}^{B_t \cap A}[\log]}$ such that $B_t \cap A \leq_{2\text{-}tt}^p S_t$. Propositions 4.13–4.18 below accomplish this task. In the following propositions, on input y to the reduction $A \leq_{2\text{-}ftt}^p S$, $g_1(y)$ and $g_2(y)$ denote the first and the second queried strings, respectively. $g(y)$ denotes the set of queried strings on input y, i.e., $\{g_1(y), g_2(y)\}$. We say that $A \leq_{2\text{-}ftt}^p S$ via g.

Proofs of the following two propositions are immediate, and thus are omitted.

Proposition 4.13 If $A \leq_{2\text{-}ftt}^p S$ via g with any of the truth-tables of Figure 2, and S is sparse, then there exists a sparse set S' in P^A such that $A \leq_{2\text{-}ftt}^p S'$ via g with the same truth-table.

Proposition 4.14 If $A \leq_{2\text{-}ftt}^p S$ via g with any of the truth-tables of Figure 3, and S is sparse, then there exists a sparse set S' in NP^A such that $A \leq_{2\text{-}ftt}^p S'$ via g with the same truth-table.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
1	0	0	0	0
2	1	1	1	1

Figure 2: Trivial truth-tables of arity two.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
3	0	0	1	1
4	1	1	0	0
5	0	1	0	1
6	1	0	1	0

Figure 3: 1-tt-related truth-tables of arity two.

Proposition 4.15 If $A \leq^p_{2\text{-}ftt} S$ via g with any of the truth-tables of Figure 4, and S is sparse, then there exists a sparse set S' in NP^A such that $A \leq^p_{2\text{-}ftt} S'$ via g with the same truth-table.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
7	0	0	1	0
8	1	1	0	1
9	0	1	0	0
10	1	0	1	1

Figure 4: Implication-related truth-tables of arity two.

Proof of Proposition 4.15: These tables are variations of the truth-table for implication. We prove for the case of *truth-table* 10. The other cases can be proved similarly.

Without loss of generality, we assume that g_1 and g_2 are honest and that for all y, $g_1(y)$ and $g_2(y)$ begin with 0 and 1, respectively. Let $S_1 = \{0x \mid (\exists y)[y \in \overline{A} \wedge 0x \in g(y)]\}$. S_1 has the strings in S that guarantee all the strings in $\overline{A}$ to be rejected. However, some of the strings in A may also get rejected if S_1 is used instead of S in the reduction. In order to remedy this problem, we add S_2 to S', where

S_2 is given by

$$S_2 = \{1x \mid (\exists x', y_1, y_2) \\ [y_1 \in \overline{A} \wedge y_2 \in A \wedge 0x' \in g(y_1) \wedge g(y_2) = \{0x', 1x\}]\}.$$

Since S_2 contains only second queries, there is no more chaining of side effects. Note that, if the first and second queries were not separated, some strings in S_2 might be used as first queries, potentially leading to a chain of side effects. Let $S' = S_1 \cup S_2$. It is easy to verify that $S' \subseteq S$ and S' satisfies the theorem. ∎

The following proposition is essentially a special case of Theorem 4.6.

Proposition 4.16 If $A \leq^p_{2\text{-}ftt} S$ via g with any of the truth-tables of Figure 5, and S is sparse, then there exists a sparse set S' in NP^A such that $A \leq^p_{2\text{-}ftt} S'$ via g with the same truth-table.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
11	1	0	0	0
12	0	1	1	1

Figure 5: Conjunctive-related truth-tables of arity two.

Proof of Proposition 4.16: These tables are variations of the truth-table for conjunction. Without loss of generality, we assume that g_1 and g_2 are honest. It is easy to see that, for the case of truth-table 11, $S' = \{x \mid (\exists y)[y \in A \wedge x \in g(y)]\}$ satisfies the condition. The other case can be proved similarly. ∎

The following proposition is essentially a special case of Theorem 4.2. We omit its proof.

Proposition 4.17 If $A \leq^p_{2\text{-}ftt} S$ via g with any of the truth-tables of Figure 6, and S is sparse, then there exists a sparse set S' in $\mathrm{P}^{\mathrm{NP}^A[\log]}$ such that $A \leq^p_{2\text{-}ftt} S'$ via g with the same truth-table.

Proposition 4.18 If $A \leq^p_{2\text{-}ftt} S$ via g with any of the truth-tables of Figure 7, and S is sparse, then there exists a sparse set S' in $\mathrm{P}^{\mathrm{NP}^A[\log]}$ such that $A \leq^p_{2\text{-}ftt} S'$ via g with the same truth-table.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
13	1	1	1	0
14	0	0	0	1

Figure 6: Disjunctive-related truth-tables of arity two.

Table Number	First Query Answered **yes**		First Query Answered **no**	
	2nd Ans. **yes**	2nd Ans. **no**	2nd Ans. **yes**	2nd Ans. **no**
15	0	1	1	0
16	1	0	0	1

Figure 7: Exclusive-or-related truth-tables of arity two.

Proof of Proposition 4.18: These tables are variations of the truth-table for exclusive-or. We prove for the case of *truth-table* 15. The other case can be proved similarly.

We will show that, given a string x, we can determine whether $x \in S'$ for some unambiguously determined S', which is a well-behaved approximation to S. Although we cannot guarantee $S' \subseteq S$, we can make sure that S' is sparse and satisfies the requirements of the proposition.

Without loss of generality, assume that there exists a polynomial-time computable and invertible one-to-one function $h : 0^* \to 0^*$ such that for all $y \in \Sigma^*$, it holds that $0^{|g_1(y)|} = 0^{|g_2(y)|} = h(0^{|y|})$. (This condition makes the sparseness of S' very simple to argue.) Let $D_x = \{y \in \Sigma^* \mid 0^{|y|} = h^{-1}(0^{|x|})\}$, i.e., the set of strings, when reduced, might query x. Let $g(A) = \bigcup_{y \in A} g(y)$. Clearly, $||g(D_x) \cap S||$ is bounded by a polynomial in $|x|$; let $s(|x|)$ denote this polynomial bound.

Let $G_1 = \{g(y) \mid y \in D_x \cap A\}$, $G_2 = \{g(y) \mid y \in D_x \cap \overline{A}\}$, and $G = G_1 \cup G_2$. The problem can be easily visualized using the graph defined by the set of edges G. G consists of two different types of edges: exclusive-or type and coexistence type. The edges in G_1 are of exclusive-or type: of the two nodes of each edge in G_1, one is in S while the other is in $\overline{S}$. The edges in G_2 are of coexistence type: the two nodes of each edge in G_2 either both are in S or both are in $\overline{S}$.

Suppose that a sparse set S' is chosen, and that all the elements in S' are given the same color while those in $\overline{S'}$ are given another color.

It is easy to see that, in order to satisfy the requirement of the above proposition, it suffices to show that the two endpoints of each edge in G_1 are colored differently while those of each edge in G_2 have the same color. Thus, the problem of choosing S' can be considered as a two-coloring problem in which the sets G_1 and G_2 has to be preserved. We now describe a $\mathrm{P}^{\mathrm{NP}^A[\log]}$ algorithm to choose such S'.

(1) Check if there exists a connected component of G that contains x and at least $2s(|x|)$ other nodes. This can be checked by asking a single query (with $m = 2s(|x|) + 1$) to an NP^A oracle defined by the following machine description.

Machine M^A on input $\langle 0^m, x\rangle$
Guess a graph G' with m distinct vertices.
If G' contains x, and forms a connected subgraph of G, then accept.

If such a connected component (G') does not exist, proceed to step (2). Otherwise, accept x if and only if $x \in S$. This can be checked by asking a single query to an NP^A oracle similar to the above one since $x \in S$ if and only if x has the less populous color in a two-coloring of G'.

(2) Find G_L, the set of edges in the maximal connected subgraph of G that contains x. Obviously, G_L is unique. Moreover, since G_L has no more than $2s(|x|)$ nodes, it has no more than $s(|x|)(2s(|x|) - 1)$ edges. Hence, G_L can be found along some computation path of an NP^A machine, once $||G_L||$ is known. Accept x if and only if x has the less populous color in a two-coloring of G_L.

Note that $||G_L||$ can be obtained by a binary search using an NP^A oracle defined by the following machine description.

Machine $M^A_{G_L}$ on input $\langle 0^m, x\rangle$
Guess a graph G' with m distinct edges.
If G' contains x, and forms a connected subgraph of G, then accept.

Clearly, both the steps (1) and (2) maintain $||S'^{=|x|}|| \leq ||S^{=|x|}||$, thereby keeping S' sparse. It is easy to see that S', via the reduction

g, preserves G_1 and G_2; this proves the correctness of the algorithm. It is not hard to fill in the details of the algorithm in a way that guarantees that $S' \in \mathrm{P}^{\mathrm{NP}^A[\log]}$. ∎

The above Propositions 4.13–4.18, together with Proposition 4.11 and Proposition 4.12, prove Theorem 4.3. ∎

5 Low Instance Complexity and Polynomial-Time Reductions

Instance complexity, as defined by Ko, Orponen, Schöning, and Watanabe [KOSW86,Orp90,KOSW90], is a notion of the complexity of specific instances of a problem—a topic that standard complexity theory is ill-suited to study (as any single instance is trivial). In this paper, we are primarily concerned with sets of "low" instance complexity: IC[log,poly] (introduced in [KOSW86]).

Definition 5.1 We say that a set A is in IC[log,poly] if there exist a constant $c > 0$, a polynomial t and a set $\Pi \subseteq \Sigma^*$ of programs[8] such that for every $x \in \Sigma^*$

1. there exists a $p \in \Pi^{\leq c \log(|x|)+c}$ such that p decides x in time $t(|x|)$ according to A, and

2. for every $p \in \Pi$ it holds that if p decides x in time $t(|x|)$ then p decides x according to A.

We show that the sets of low instance complexity are intimately related to the study of reductions to sets of low information content. In particular, a set A is of low instance complexity if and only if it both conjunctively and disjunctively reduces to tally sets (say T_0 and T_1, respectively). Note that this implies that A reduces disjunctively and conjunctively to the single tally set: $\{0^{2i} \mid 0^i \in T_0\} \cup \{0^{2i+1} \mid 0^i \in T_1\}$.

Theorem 5.2 A is in IC[log,poly] if and only if there exist tally sets T_0 and T_1 such that $A \leq_c^p T_0$ and $A \leq_d^p T_1$.

[8] For a fixed efficient universal machine. In fact, "p decides x in time $\cdots$" in this definition refers to the run of the fixed universal machine on $\langle p, x\rangle$. We refer the reader to [Orp90, p. 21] for details of the universal machine scheme.

Proof of Theorem 5.2:
For $A \in$ IC[log,poly], let c be a constant, t be a polynomial and Π be a set of programs as in Definition 5.1. We denote by $ord(p)$ the position of the string p in the lexicographical enumeration of Σ^*. We can encode Π into the tally set $T = \{0^{ord(p)} \mid p \in \Pi\}$. Let g be a truth-table condition generator that on input x computes the disjunction of the encodings $0^{ord(p)}$ of length less than or equal to $c\log(|x|)+c$ for all programs p that accept x in time $t(|x|)$. Then $A \leq_d^p T$ via g since x is in A if and only if there exists a program in $\Pi^{\leq c\log(|x|)+c}$ that accepts x in time $t(|x|)$.

Since IC[log,poly] is closed under complementation, this proves also the inclusion IC[log,poly] $\subseteq \mathrm{R}_c^p$(TALLY).

For the reverse inclusion, let $A \leq_c^p C$ and $A \leq_d^p D$ where C and D are tally sets, and g_c and g_d are the respective polynomial-time truth-table condition generators. We assume that all generated queries are in 0^*. For every $0^i \in 0^*$ it is easy to construct a program p_i^c that on input x computes $g_c(x) = y_1 \wedge \ldots \wedge y_m$ and rejects if $0^i \in \{y_j \mid 1 \leq j \leq m\}$. Otherwise p_i^c goes into an infinite loop. It is clear that the running time of p_i^c is polynomially bounded on all inputs that it rejects, and that the size of p_i^c is $\mathcal{O}(\log(i))$. For every $x \notin A$ the conjunction $g_c(x)$ contains a query $0^j \notin C$. Thus for every $x \notin A$ there exists an index j, polynomially bounded in $|x|$, such that $0^j \notin C$ and p_j^c on input x rejects.

Similarly, for every $0^i \in 0^*$ there is a program p_i^d that on input x computes $g_d(x) = z_1 \vee \ldots \vee z_m$ and accepts if $0^i \in \{z_j \mid 1 \leq j \leq m\}$. Otherwise it goes into an infinite loop. The programs p_i^d are also $\mathcal{O}(\log(i))$ in size and have polynomial running time on all inputs that are accepted. For every $x \in A$ the disjunction $g_d(x)$ contains a query $0^j \in D$. Thus for every $x \in A$ there exists an index j, polynomially bounded in $|x|$, such that $0^j \in D$ and p_j^d on input x accepts.

Hence, taking $\Pi = \{p_j^d \mid 0^j \in D\} \cup \{p_j^c \mid 0^j \notin C\}$ as the set of programs, it follows that $A \in$ IC[log,poly]. ∎

Using the above characterization of IC[log,poly] it is easy to see that IC[log,poly] is closed under bounded truth-table reductions (a different proof that explicitly constructs programs appears in [KOSW90]):

Theorem 5.3 [KOSW90] IC[log,poly] is closed downward under $\leq_b^p$-reductions.

Proof of Theorem 5.3:
It is easy to see that $\mathrm{coR}_d^p(\mathrm{TALLY}) = \mathrm{R}_c^p(\mathrm{TALLY})$, which immediately implies the closure of $\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})$ under complementation. In fact, for every set A in $\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})$ there exists a single tally set T such that A and $\overline{A}$ are in $\mathrm{R}_d^p(T) \cap \mathrm{R}_c^p(T)$ and thus $\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})$ is also closed under one truth-table reductions.

The second observation is that $\mathrm{R}_{bc}^p(\mathrm{R}_d^p(\mathrm{TALLY})) \subseteq \mathrm{R}_d^p(\mathrm{R}_{bc}^p(\mathrm{TALLY})) \subseteq \mathrm{R}_d^p(\mathrm{R}_m^p(\mathrm{TALLY})) \subseteq \mathrm{R}_d^p(\mathrm{TALLY})$. Here we use that $\mathrm{R}_{bc}^p(\mathrm{TALLY}) = \mathrm{R}_m^p(\mathrm{TALLY})$ (see [Ko89]).

Similarly, $\mathrm{R}_{bd}^p(\mathrm{R}_c^p(\mathrm{TALLY})) \subseteq \mathrm{R}_c^p(\mathrm{TALLY})$, and thus it follows that $\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})$ is closed under bounded conjunctive and bounded disjunctive reducibilities. Combining this, we get

$$\begin{aligned}
\mathrm{R}_b^p(\mathrm{IC[log,poly]}) &= \mathrm{R}_b^p(\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})) \\
&\subseteq \mathrm{R}_{bc}^p(\mathrm{R}_{bd}^p(\mathrm{R}_{1-tt}^p(\mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY})))) \\
&\subseteq \mathrm{R}_d^p(\mathrm{TALLY}) \cap \mathrm{R}_c^p(\mathrm{TALLY}) = \mathrm{IC[log,poly]}. \quad \blacksquare
\end{aligned}$$

Corollary 5.4 [KOSW90] If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_b^p$-hard for NP, then $A \notin \mathrm{IC[log,poly]}$.

Proof of Corollary 5.4:
Suppose that a bounded truth-table hard set for NP is in IC[log,poly]. It follows by Theorem 5.3 that $\mathrm{NP} \subseteq \mathrm{IC[log,poly]}$. From the result of [KOSW86] that if a set of low instance complexity is many-one hard for NP then P = NP, it follows that P = NP. ∎

Using Theorem 5.2 and the fact that NP has neither $\leq_c^p$-hard tally sets nor $\leq_d^p$-hard tally sets unless P = NP, it follows that NP has neither $\leq_c^p$-hard sets nor $\leq_d^p$-hard sets in IC[log,poly] unless P = NP.

Corollary 5.5 If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_d^p$-hard for NP, then $A \notin \mathrm{IC[log,poly]}$.

Proof of Corollary 5.5:
Suppose that a disjunctive truth-table hard set for NP is in IC[log,poly]. Since $\mathrm{R}_d^p(\mathrm{IC[log,poly]}) \subseteq \mathrm{R}_d^p(\mathrm{TALLY}) \subseteq \mathrm{R}_d^p(\mathrm{coSPARSE})$, it follows from Ukkonen's result [Ukk83] that P = NP. ∎

Corollary 5.6 If $\mathrm{P} \neq \mathrm{NP}$ and A is $\leq_c^p$-hard for NP, then $A \notin \mathrm{IC}[\log, \mathrm{poly}]$.

Proof of Corollary 5.6:
Suppose that a conjunctive truth-table hard set for NP is in IC[log,poly]. Since $\mathrm{R}_c^p(\mathrm{IC[log,poly]}) \subseteq \mathrm{R}_c^p(\mathrm{SPARSE})$, it follows from Corollary 3.4 that $\mathrm{P} = \mathrm{NP}$. ∎

Finally, using Theorem 3.13 and the fact that every tally set is in $\mathrm{IC}[\log, \mathrm{poly}]$, we can conclude that for truth-tables of size $\omega(\log n)$, no analog of Corollary 5.4 can be proven by any relativizable proof technique.

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